WHAT'S THIS GEOMETRIC ALGEBRA ALL ABOUT?

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1 INTRODUCTION

There is a folk theorem that circulates among some of my friends that might loosely be called the "Conservation of Work Theorem". It holds that no matter how elegant and succinct is the mathematics in which a concept is described, it always takes the same amount of work to obtain a number from the concept.

Hestenes has proposed, in his Oersted Medal Lecture 2002 [1] to reform the mathematical language of Physics by replacing the customary use of vectors, with their associated operation with what he calls "Geometric Algebra" (abbreviated as "GA"). He argues that "the fundamental geometric concept of a vector as a directed magnitude is not adequately represented in standard mathematics...we need multiplication rules that enable us to compare directions and magnitudes of different vectors."[2] He then describes his GA as based upon Clifford Algebra.[3].

Hesthenes' lecture is unfortunately devoid of examples of applications to elementary problems in classical physics. For that, however, we can turn to the lecture notes from the course being taught at to advanced undergraduates at Cambridge University.[4]. I have chosen, for purpose of comparison with conventional methods, the Kepler problem. I shall sketch solutions of the problem in three different ways: firstly, a very pedestrian approach in a spherical coordinate system, then a coordinate-independent approach more or less following the derivation in The Mechanical Universe, Chapter 26, and, finally, as done in the Cambridge lectures.

2 KEPLER FOR A COURSE IN VECTOR ANALYSIS

2.1 GIBBS WITH COORDINATES

Define spherical coordinates r, θ, ϕ, a vector $\vec{R}$, and unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}, \hat{r}$ in a rectangular coordinate system according to the relations:

$$\vec{R} = r\hat{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \;\; z = r\cos\theta \;\; x = r\sin\theta\cos\phi \;\; y = r\sin\theta\sin\phi \tag{1}$$

In order to do dynamics we need the velocity vector

$$\dot{\vec{R}} = \dot{r}\hat{r} + r\dot{\hat{r}} = \vec{v_r} + \vec{v_\theta} + \vec{v_\phi} \tag{2}$$

which leads us to the two unit vectors orthogonal to $\hat{r}$, namely

$$\hat{\theta} = \hat{\imath}\cos\theta\cos\phi + \hat{\jmath}\cos\theta\sin\phi - \hat{k}\sin\theta \;\; and \;\; \sin^2\theta\hat{\phi} = \sin^2\theta(-\hat{\imath}\sin\phi + \hat{\jmath}\cos\phi) \tag{3}$$

Note that the cross products among the unit vectors satisfy $\hat{r} \times \hat{\theta} = \hat{\phi}$ so that these unit vectors define an orthogonal right-handed coordinate system.

We finally, after all this work, obtain the components of the acceleration vector:

$$\ddot{\vec{R}} = \ddot{r}\hat{r} + 2\dot{r}\dot{\hat{r}} + r\ddot{\hat{r}} = a_r\hat{r} + a_\theta\hat{\theta} + a_\phi\hat{\phi} \tag{4}$$

with

$$\begin{aligned} a_r &= \ddot{r} - r(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) \\ a_\theta &= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r(\dot{\phi}^2 \sin\theta\cos\theta) \\ a_\phi &= \frac{1}{r\sin\theta}\frac{d}{dt}(r^2\sin^2\theta\dot{\phi}) \end{aligned} \tag{5}$$

The last set of equations simplify greatly when we realize that we can align the coordinates so that at some instant of time we can make θ equal to $\frac{\pi}{2}$ and $\dot{\theta} = 0$. Then the second of these last equations tells us that the orientation of the polar axis does not change; the trajectory is a plane. This is the starting point in most elementary textbooks. We see already, however, that the third equation gives a conserved quantity for central forces (forces having no ϕ- component). The conserved quantity is just $\vec{R} \times \dot{\vec{R}}$ so we can recognize it as the angular momentum per unit mass and symbolize it as a vector $\vec{l}$ having magnitude l.

The component equations then assume their familiar form:

$$\begin{aligned} a_r &= \ddot{r} - \frac{l^2}{r^2} \\ a_\theta &= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \\ a_\phi &= \frac{1}{r}\frac{d}{dt}(r^2\dot{\phi}) \equiv \frac{1}{r}\frac{dl}{dt} \end{aligned} \tag{6}$$

with the last two equations being equal to zero when the force is central. The acceleration vector then has just two components: $a_r \ \ and \ \ a_\phi$.

2.2 GIBBS WITHOUT COORDINATES

The hard labor in the previous subsection was unnececesary. Let's just start with the observation that $\hat{r} \cdot \hat{r} = 1$ The time derivative of the last equation gives $2\hat{r} \cdot \dot{\hat{r}} = 0$; that is, $\hat{r}$ and $\dot{\hat{r}}$ are orthogonal vectors.

The angular momentum per unit mass $\vec{l} = \vec{R} \times \dot{\vec{R}}$ can be obtained immediately, using the first equality of Eq. 2, as

$$\vec{l} = r^2\hat{r} \times \dot{\hat{r}} \tag{7}$$

Eq. 7 tells us that the three vectors $\hat{r}$, $\dot{\hat{r}}$, and $\vec{l}$ are mutually orthogonal.

To see that $\vec{l}$ is conserved in the presence of a central force we turn to the first equality in Eq 4, $\ddot{\vec{R}} = \ddot{r}\hat{r} + 2\dot{r}\dot{\hat{r}} + r\ddot{\hat{r}}$, which is equal to the force (divided by the mass). If the force is central then its cross product with $\hat{r}$ is zero. That is,

$$\hat{r} \times \ddot{\vec{R}} = 2\dot{r}\hat{r} \times \dot{\hat{r}} + r\hat{r} \times \ddot{\hat{r}} = 0 \tag{8}$$

But the right hand side of the first equality in Eq. 8 is just $\frac{1}{r}$ times the time derivative of $\vec{l}$ which must, accordingly, vanish.

The gravitational force is proportional to r^{-2}. It turns out that for this special central force there is another constant of the motion that greatly simplifies the solution of the Kepler problem. In order to see this we can write Newton's third law for such a force in the form $\frac{d}{dt}\dot{\vec{R}} = -Kr^{-2}\hat{r}$, where K is some constant. Taking the cross-product with the constant vector $\vec{l}$ gives us that

$$\frac{d}{dt}(\vec{l} \times \dot{\vec{R}}) = -Kr^{-2}\vec{l} \times \hat{r} = -K(\hat{r} \times \dot{\hat{r}}) \times \hat{r} = -K\frac{d}{dt}\hat{r} \tag{9}$$

where, in the last equality we had to know how to expand the vector triple product. We conclude from the last equation that $\vec{l} \times \dot{\vec{R}} + K\hat{r}$ is a conserved vector that we can call $K\vec{\epsilon}$. This constant fixes the shape of the orbit in the plane perpendicular to $\vec{l}$ (the ellipse does not precess).

The equation for $K\epsilon$ is, in fact, the solution of the Kepler problem although we'll have to do a bit more work to make this fact obvious.

Use the equation for the square of the cross-product of two perpendicular vectors to see that $|\vec{l}|^2 = r^4|\dot{\hat{r}}|^2$. Since $\vec{l}$ is a constant vector, we can write that $\dot{\hat{r}} = \frac{l}{r^2}\hat{\phi}$ where

$$l\dot{r}\hat{\phi} + \frac{1}{r^2}(K - \frac{l^2}{r})\hat{r} = -K\vec{\epsilon} \tag{10}$$

Writing $\vec{\epsilon} = \epsilon\hat{\phi}$, where I have chosen the direction of epsilon to be tangent to the orbit at a point where $\dot{r} = 0$, and taking the dot product of Eq. 10 with the radial unit vector gives the final result that

$$\frac{1}{r} = (1 + \epsilon\cos\phi)Kl^{-2} \tag{11}$$

where ϕ is the angle between the radius vector and the direction of the fixed vector $\vec{\epsilon}$. Eq. 11 is the equation of an ellipse when ϵ is less than unity.

3 KEPLER in GA according to Cambridge

I will speak of a "Gibbsian vector" when I mean something with three components, as defined by Gibbs, which we represent by an arrow (length and direction) in three dimensions. I'll make connection with GA by defining different kinds of objects that I will relate to Gibbsian vectors. This is pretty much what Hestenes does in his Oersted medal lecture. The objects that I introduce are denoted by the symbols e_i *and* $\hat{1}$. If you are mathematically sophisticated I will tell you that these objects are the elements of a vector space over the real numbers. If you've had a course in linear algebra I will tell you that these objects are something like matrices. If you're in neither category I don't know what to tell you.

I now make the vector space into an algebra by defining multiplication among the elements. The rules are, there are n elements e_i:

$(R1)\ e_i^2 = -\hat{1}\ i = 1, .., n$
$(R2)\ e_i e_j = -e_j e_i\ i \neq j$
$(R3)\ e_i \hat{1} = \hat{1} e_i\ =\ e_i\ i = 1, .., n$
ALSO:
(R4) Multiplication is associative, and
(R5) All of the ordered products of the e_i are basis elements of the vector space. Now let's see where this gets us.
(n=1)
Every element of the vector space can be written as $x\hat{1} + ye_1$ where x, y are real numbers. The product of two such elements with respective real number pairs (x,y) and (u,v) is $(xu - yv)\hat{1} + e_1(xv + yu)$. The coefficients of $\hat{1}$ and e_1 are exactly what we would get for the real and imaginary parts if we were multiplying complex numbers. A devout mathematician would tell you that we have defined a vector space over a Clifford algebra that is isomorphic to the algebra of complex numbers.

Four remarks: (1) Note that a complex number can represent a vector in a two-dimensional space. It makes sense, therefore to think of the elements of the space as vectors in the Gibbsian sense.

(2) The identification with Gibbsian vectors fails unless we have multiplication that gives a positive number as the length of a vector. But we know from complex analysis how to deal with this issue. The length of a vector (squared) is not the square of the vector but the product with its complex conjugate. The isomorphism of complex conjugation is the transformation from e_1 to $-e_1$.

(3) It is often easier for the novice to deal with abstract entities like non-commuting symbols by finding matrix representations of the symbols. This is easily done for e_1 and $\hat{1}$ with the $2x2$ matrix representation:

$$\hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{12}$$

(4) Note the demands that we are putting on the student who is being newly introduced to Clifford algebras. The student must not only deal with abstract symbols (the college juniors in the last class I taught were still struggling with high school algebra after completing calculus-based physics courses), but with vector spaces, complex numbers, and non-commuting symbols. We've imported all that baggage just to have a fancy way to talk about complex numbers.
(n=2)
Now our vector space has 4 elements, namely
$\hat{1}, e_1, e_2\ and\ e_1 e_2$. Let's label these respectively $\sigma_1, \sigma_2, \sigma_3$. Note that their products satisfy:

$$\sigma_i \sigma_j = \sigma_k\ \ (i, j, k) = (1, 2, 3) cyclic \tag{13}$$

and each pair anti-commutes. There is a $2x2$ matrix representation of the $\sigma' s$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\quad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\quad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

If $\vec{V}$ is any Gibbs vector, with components $x^i\ \ i = 1, 2, 3$ and s is any real number, then we can write every element of the vector space as $s\hat{1} + x^i \sigma_i$ (summed on repeated indices). In a matrix representation this will look like:

$$s\hat{1} + x^i \sigma_i = \begin{pmatrix} s + ix^3 & x^1 + ix^2 \\ x^1 - ix^2 & s + ix^3 \end{pmatrix} \tag{14}$$

It is in this sense and only in this sense (or some equivalent) that the GA permits one to add scalars to vectors.

What I have defined for you here is called "the quaternion algebra". If I write the "scalar product" of the Gibbs vector $\vec{V}$ with the three σ's (treated as the three components of a vector) as $\vec{V} \cdot \vec{\sigma}$ then, following Hestenes, I will call every such scalar product a "vector" V.

The algebra of the σ's can be concisely summarized with the formula $\sigma_i\sigma_j = -\delta_{ij} + \epsilon_{ij}^k\sigma_k$. This leads to the formula for the product of two vectors (using the new definition of "vector") $uv = -\vec{u} \cdot \vec{v}\dot{1} + -\vec{u} \times \vec{v} \cdot \vec{\sigma}$.

We need one more bit of notation and then we can return to the Kepler problem. The last equation combines the dot- and cross- products of the Gibbs vectors, It is convenient to separate them by noting that any product of two elements may be written as the sum of its symmetric and antisymmetric products. So in the quaternion algebra, we can define the equivalent of the Gibbs dot product by $(u|v) = -(uv + vu)/2$ and the Gibbs cross product is contained in the so-called "wedge product" $u \wedge v = (uv - vu)/2$. Obviously the vector product $uv = -(u|v) + u \wedge v$.

The definition of the wedge product doesn't yet tell you how the handle the wedge product of more than two vectors. That issue is determined by the requirement that the Clifford algebras are associative. The wedge product of n vectors is then $\frac{1}{n!}$ times the completely antisymmetrized sum of the $n!$ orderings of the vectors. Obviously, with the quaternion algebra (3-dimensions) there can be no more than three vectors in such a product. The wedge product of three vectors is the isomorph of the scalar triple product ($\vec{A} \cdot \vec{B} \times \vec{C}$) of Gibbs vectors.

Now for the Cambridge treatment of the Kepler problem. As before, we start with Newton's second law. In order to distinguish the vectors from the Gibbs vectors in the preceding section we'll use a vector $\mathbf{x} = \vec{R} \cdot \vec{\sigma}$. Newton's law then becomes $m\ddot{\mathbf{x}} = \mathbf{f}$ where $\mathbf{f}$ is the force vector (the Gibbs vector dotted into sigma). It is only a notational change to write the angular momentum per unit mass as a wedge product, selecting out the cross-product part of $\mathbf{x}\dot{\mathbf{x}}$.

Just as with Gibbs, we write r as the magnitude of the vector $\mathbf{x}$ so that $r^2 = (\mathbf{x}|\mathbf{x}) = -\mathbf{x}\mathbf{x}$ and $\mathbf{x} = r\hat{\mathbf{x}}$. Then Eq. 2 becomes, in our new language:

$$\dot{\mathbf{x}} = \frac{d}{dt}(r\hat{\mathbf{x}}) = \dot{r}\mathbf{x} + r\dot{\hat{\mathbf{x}}} \tag{15}$$

The angular momentum $\mathbf{l}$ becomes

$$\mathbf{l} = \mathbf{x} \wedge (\dot{r}\hat{\mathbf{x}} + r\dot{\hat{\mathbf{x}}}) = r^2\hat{\mathbf{x}}\dot{\hat{\mathbf{x}}} \tag{16}$$

where we are able to drop the "wedge" symbol in the last term because the two vectors in the product are orthogonal. For orthogonal vectors the product of vectors is essentially identical to the cross-product of Gibbs vectors.

The next step is quite analogous to the way we proceeded with Gibbs vectors. Multiply Newton's law for a $\frac{1}{r^2}$ force by $\mathbf{l}$ to get

$$\mathbf{l}\ddot{\mathbf{x}} = -\frac{K}{mr^2}\mathbf{l}\hat{\mathbf{x}} = -K\hat{\mathbf{x}}\dot{\hat{\mathbf{x}}}\hat{\mathbf{x}} = K\dot{\hat{\mathbf{x}}} \quad and \quad \frac{d}{dt}\mathbf{l} = 0$$

so that

$$\frac{d}{dt}(\mathbf{l}\dot{\mathbf{x}} - K\hat{\mathbf{x}}) = 0 \tag{17}$$

This gives the equation for the orbit as

$$\mathbf{l}\dot{\mathbf{x}} = K(\hat{\mathbf{x}} + \epsilon) \tag{18}$$

where ϵ is the fixed vector that we discovered in the preceding section. (It is known in the literature as the Lenz-Runge vector). Multiply the last equation by $\mathbf{x}$ and take the scalar part to get the same equation for the orbit as Eq. 9.

4 Conclusion

I fail to see either simplification or enlightenment from reviving the quaternion approach to elementary 3-d dynamics. In higher dimensions, or in the context of much more complicated problems there might be some justification for a group theoretical approach, although Clifford algebras do not seem to lie in the favored direction among the sophisticated practitioners (see *e. g.* Arnold's text on mechanics).

After completion of an early version of this exposition I discovered that the September 2002 issue of AJP is devoted to algebraic mechanics.

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References

[1] David Hestenes,

[2] Reference 1, Sect. 5

[3] Reference 1, Sect. VIIIA

[4] http://www.mrao.cam.ac.uk/ clifford/ptIIIcourse