# CONCEPTUAL CALCULUS

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# Contents

# OVERVIEW

(for students, not to be read by teachers)

There is good news and bad news about this textbook, which we think is quite different from many of the textbooks that you are used to. Look in the back of the book. You will quickly see that there are no answers to problems. You may take that fact as the bad news.

The good news is that the correct answer to each problem is given, or at least suggested, in the text following the problem. That arrangement saves you the time and trouble of finding your way to the back of the book to see if you have answered a problem correctly. It also shows you that we intend for you to read the text.

Many textbooks give you examples that you can copy in order to do the homework problems. We think that we have a better idea. We guide you through many of the problems so that you are, in effect, writing your own examples. When you get stuck, or make a mistake, you can then discuss with your teacher what it is that you got stuck on. Your teacher is then not only teaching you calculus but, more importantly, is helping you to learn how to learn.

Don't forget. When you have worked your way out of getting stuck, or corrected a mistake, you have probably learned something new!

Calculus is the study of how things change. Better yet, it is the study of how to talk about change.

We often talk in real life about things changing and how fast things change. "My car accelerates from rest to 40 miles an hour in 5 seconds" is a remark about change. "The world record for the 1 mile run dropped from 3:59.4 to 3:46.32 between 1954 and 1985" is a statement about change. But these statements do not give enough detail to suggest why things change the way they do.

Take the case of the accelerating automobile, for example. If the velocity at the end of 5 seconds was 40 miles per hour, does that mean that the velocity at the end of 2 seconds was 2/5 of 40 or 16 miles per hour? The answer turns out to be "Not necessarily." Also, at the end of 10 seconds will the car be going 80 miles per hour? The answer is "Probably not."

Calculus helps us to understand these answers. It also helps us, in many cases, to find the correct answers.

Here are some questions that you should be able to answer at the end of this course.

1. My bank pays 5% compound interest, compounded *continuously*, on my savings account. What interest rate would I need at a bank that compounds its interest *monthly* in order to have the same amount of savings in the long run?

2. A sheet of metal 12 inches wide is to be made into a gutter by turning up the edges at right angles to the base. What should be the width of the gutter to maximize the volume the gutter can carry?

3. The material used in making the tops and bottoms of certain soft drink cans is twice as expensive as the material for the sides. The cans are to contain a fixed volume of liquid. What is the ratio of height to radius for minimum cost if the cans are right circular cylinders? Does the ratio depend on the volume?

## NOTES FOR TEACHERS
(not to be read by students)

Existing calculus textbooks seem to assume that a student learns the subject in a logically linear fashion. First we learn about functions and continuity, then about limits, then the derivative, and so forth. The student doesn't learn, until somewhere around chapter 4, that all of this mathematical apparatus is somehow related to solving problems concerning motion, velocity, acceleration, and related concepts.

Few modern undergraduate mathematics texts, in fact, clearly recognize that most mathematics was invented for the purpose of making it easier to solve practical problems.

We think that this text is different in at least two respects. The structure of the text is intended to implement our recognition that the learning of mathematics is a highly non-linear process. Oftentimes the mathematical (or "logical") justification for a procedure cannot be understood by a student until long after the student has learned the procedure. All of us, we think, gain new insights into most topics in mathematics each time we revisit the topic. There is need, therefore, for texts that emphasize such revisitations as a part of the learning process. Our emphasis on the non-linearity of learning is the first of the differences.

The second difference from most texts[1] is the emphasis, from the very beginning, on applications of calculus to the solution of practical problems. We believe that many students complete a year's course in calculus without having a clear idea of any practical uses of the subject. (The same is true, unfortunately, of most courses in elementary algebra.)

Practical applications aside, for the moment, there are, in our view, two principal mathematical themes for a course in introductory calculus: the derivative measures the slope of a curve at a point, and the definite integral measure the area under a curve between the two points. We think that these two themes are conceptually within every student's grasp, and we have tried to write this text to echo these themes clearly and unmistakably.

Many students taking this course will be weak on algebra skills and will have had minimal exposure to the concept of "proof". The design of this text is intended to assist the teacher to address such weaknesses. The general design of the course is intended to incorporate the pedagogical principle set out in Arnold Arons' book *A Guide to Introductory Physics Teaching* (Wiley, New York 1990). Although Arons' book ostensibly addresses the teaching of physics, much of the subject matter addresses means for remedying the mathematical deficiencies of typical engineering and science students.

[1]a notable exception, in this respect, is Downing, *Calculus the Easy Way*, (Barrons, Hauppage, N.Y., 1996)

We have tried to avoid many of the proofs that are found in typical calculus texts because they are not, for the most part, read and understood by students. Such proofs as are included are intended to be learned by the students sufficiently well to be reproduced from memory. We have similarly avoided the early introduction of concepts that, while important to the rigorous development of the subject of calculus, are totally incomprehensible to beginning students. Consider, as an example, Professor Rucker's comment in *Infinity and the Mind* (Birkhäuser, Boston 1982), p. 87:

> As someone who has spent a good portion of his adult life teaching calculus courses for a living, I can tell you how weary one gets of trying to explain the complex and fiddling theory of limits to wave after wave of uncomprehending freshmen.

Our approach is, instead, intended to build on students' intuitive concepts, with mathematical rigor generally left for later courses where rigor aids, instead of detracts from, the conceptual development. References will be given, however, for all important rigorous derivations.

This text, accordingly, incorporates some so-called 'non-standard' methods. That is, we allow ourselves, when appropriate (for example in treating difference quotients in finding the slope of a curve), to treat an extremely small increment of the independent variable (which we have dubbed the 'dibbl') as if its square were equal to zero, thus giving us the opportunity to exercise students on the powers of large and small numbers, leading to a single new concept that the student must confront. This approach, which allows us to defer the notion of limits, has an honorable history going back to Abraham Robinson's 'non-standard analysis' and infinitesimals and even before. No mathematician would have the slightest difficulty in translating what we say into conventional limit notation.

Existing texts that incorporate non-standard analysis [2] have failed, in our view, to clarify the subject because the authors apparently felt honor-bound to clutter their presentation of the intuitively accessible ideas by first presenting rigorous justifications of the mathematics. We contend that the basic ideas are easily understood by students who are probably totally impervious to the justifications. We also contend that students will experience far less discomfort using these dibbls than they will experience by trying to absorb and utilize the notion of limits or other rigors from the outset.

Conceptual development can be greatly aided by kinesthetic experiences. The relatively recent availability of sonic ranging devices attached to desktop computers has helped impress students of physics with the concepts of position, velocity and acceleration. Some math teachers have realized that the same exercises that physics teachers use are effectively

[2] *e.g.* H. Jerome Keisler, *Elementary Calculus, an Approach Using Infinitesimals*, Bodgen & Quigley, 1971, (2nd ed. entitled *Elementary Calculus: an Infinitesimal Approach* 1986), and Keith Stroyan's *Calculus Using Mathematica* (1993) (second edition entitled *Calculus, the Language of Change* 1997)

teaching students the mathematical concepts of function, derivative and integral. We have accordingly included some sonic ranging exercise descriptions in a second Appendix.

The book is intended to accommodate a pace of one chapter per week. Students would be expected to complete all of the exercises in each chapter, showing all work in detail. Rather than provide answers to the exercises at the back of the book, we have included answers in the textual material following the exercises. This innovation is designed to lure the students into reading the textual material and, hopefully, trying to understand it, rather than merely skimming for methods to solve the given exercises. This is one of several ways that we have attempted to send the message that this is a book to be read.

Students are overly reliant upon calculators. We recommend that calculators not be used for end-of-chapter exercises or quizzes. Calculators can be useful for certain exercises, as will be noted on occasion.

We recommend a 15 minute closed book quiz at the end of each week, and a two hour closed book final exam at the end of each semester. The weekly quiz helps students stay abreast of the course; experience shows that the students who fall behind seldom catch up. Repeated quizzes also help the teacher to "circle back" and refresh the students' understanding of previously taught material. The use of multiple short quizzes makes time consuming mid-term exams unnecessary. Sample quizzes and final exams are included in appendices at the end of the book.

We have tried to adhere, for the most part, to the American Standard Abbreviations for Scientific and Engineering Terms as reproduced, for example, in *Handbook of Chemistry and Physics* (63d Ed., CRC Press, Boca Raton, FL., 1982)

# Part I

# GETTING COMFORTABLE

# Chapter 1

# MOTION AND CHANGE

## 1.1 ZENO'S PARADOX—CAN ANYTHING MOVE?

(AN ANCIENT PUZZLE)

An object, say an arrow shot from a bow, passes through two nearby points in space as it flies through the air. Suppose that we know the distance between the two points and the time it took to get from one point to the next. Dividing the distance by the time gives the (average) velocity of the object in traveling between the two points. But what was the velocity of the arrow when it was exactly at the first point? If the velocity didn't change between the two points, then the answer is easy. But suppose that the velocity of the arrow keeps changing, as is true of most arrows. Then how can we find the velocity at any particular point?

This question was first posed about 2500 years ago by a Greek mathematician named Zeno of Elea in ancient Greece. Zeno's writings have not been found, but his argument was quoted by, among others, the ancient Greek philosopher Aristotle in his *Physics*, Book V, 9. Zeno said, in effect: a flying arrow is at a particular location at each instant of time. Since it is *somewhere* at each instant, it is therefore motionless at each and every instant. Since it is motionless at every instant, it cannot be moving.

Zeno's argument has come down to us through history under the label, "The paradox of the arrow", so we might as well refer to it using the same name. Be sure to look up the word "paradox" in your dictionary.

Zeno's argument was more than a shrewd play on words. A modern Zeno might phrase the argument this way: "If I want to talk about the velocity of an object I need to talk about pairs of points and the time it takes the object to traverse the distance between them. So if the concept 'velocity' involves pairs of points, how can I ever talk about the velocity at a single point, at least in the case where the velocity keeps changing? Because if the velocity keeps changing, then it is different at every point."

Aristotle tried to answer Zeno by arguing that time is not made up of "instants", so it makes no sense to talk about the position of the arrow at a particular instant. Aristotle reasoned that an instant in time is like a single point on a line; the point has zero width so that if we add a lot of points together we still end up with something that has zero width. Time, therefore, is no more a collection of instants than a line is a collection of points.

It must be difficult for someone brought up in modern society to understand why anyone would pay attention to Zeno's argument. We live in an age when our everyday lives are filled with measurements. Our automobiles have odometers and velocityometers that continually inform us of distance traveled and velocity of travel. Instruments that use radio and orbiting satellites help us track of our position on the surface of the earth, or in the heart of a city. Stopwatches let us measure runners' times, in athletic events, down to the nearest hundredth of a second. So what sense does it make to argue that motion cannot exist?

The subject of calculus, which is what this book is about, is the answer to Zeno's argument about the arrow. It took about 2,000 years, from Zeno to the mathematical giants of the seventeenth century (see Fig. 1-1, " History of the World in Brief"), for people to develop the ideas needed to do the calculations implied by Zeno's argument. Those ideas form the subject matter of this book.

## Exercises

(units of measure are summarized in the Appendix at the end of the book)

*The following two problems involve only skills that are learned in elementary algebra courses.*

**1.1** Two flowers are 50 cm apart. A bee flies with a constant velocity of 20 cm/sec directly from one flower to the next.

(a) How long does it take the bee to travel the distance between the two flowers?

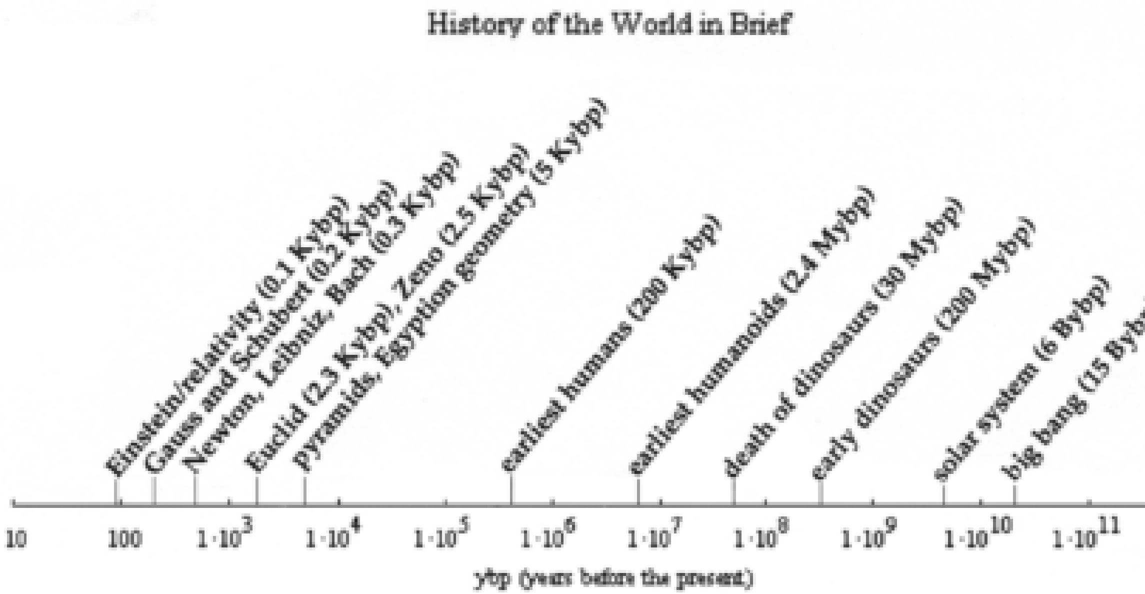
(b) If $s$ is the distance between the flowers, $v$ is how fast the bee is flying, and $t$ the time interval required for the bee's flight, write an algebraic equation that expresses $s$ in terms of $v$ and $t$. We call $v$ the bee's "velocity". The equation should be of the form $s =$?

(c) Solve the equation of part (b), for $t$. Can you use this equation to obtain the answer to part (a)?

(d) Now make a graph, using graph paper, that shows elapsed time on the horizontal ($x-$)axis and distance traveled on the vertical ($y-$)axis. The vertical axis scale should reach from 0 to 50 cm. The smallest interval on the horizontal axis should show 0.1 seconds. Can you show on this graph the position of the bee at each instant of time during its flight? If you can, do so, and save this graph for later use.

**1.2** Another bee starts from rest on the first flower, but this bee's velocity $v(t)$ increases according to the formula

$$v(t) = at$$



History of the World in Brief
Einstein/relativity (0.1 Kybp)
Gauss and Schubert (0.2 Kybp)
Newton, Leibniz, Bach (0.3 Kybp)
Euclid (2.3 Kybp), Zeno (2.5 Kybp)
pyramids, Egyptian geometry (5 Kybp)
earliest humans (200 Kybp)
earliest humanoids (2.4 Mybp)
death of dinosaurs (30 Mybp)
early dinosaurs (200 Mybp)
solar system (6 Bybp)
big bang (15 Bybp)
10
100
1·10^3
1·10^4
1·10^5
1·10^6
1·10^7
1·10^8
1·10^9
1·10^10
1·10^11
ybp (years before the present)


1.3

where $a$ is some constant and $t$ is the time traveled. We say in this case that the velocity is proportional to the time traveled and a is the constant of proportionality. This bee travels the distance between the two flowers in two seconds.

(a) What is the second bee's velocity at the instant that it reaches the second flower?

(b) What is the value of the constant of proportionality between the bee's velocity and the time traveled?

(c) Now make a second graph that shows the elapsed time on the horizontal axis and the velocity of the bee on the vertical axis? Show on this graph how fast the bee is flying at each instant of time between 0 and 2 seconds.

**1.3** Write a short paragraph (two or three complete sentences) explaining why you agree or disagree with Aristotle's criticism of Zeno's paradox of the arrow. You will have a chance, later in the book, to compare your answer with our own.

What is there about the graph in the first problem that tells us the velocity with which the bee flies? We know, since the velocity is constant, it must be the distance traveled divided by the time, or, using symbols:

$$v = \frac{s}{t} \tag{1.1}$$

But $s$ is the interval on the vertical axis between the beginning and end of the flight, while $t$ is the corresponding interval on the horizontal axis. That is, the ratio $s/t$ is just the rise over the run of the line that represents the bee's motion. But you learned in algebra and trigonometry that the rise over the run of a line on a graph is just the slope of the line, that is, it is the tangent of the angle between the line and the horizontal axis.

## 1.2 ZENO & ARISTOTLE DIDN'T KNOW ABOUT FUNCTIONS

Let's now think about a third bee that starts from rest on the first flower. This bee's velocity is such that the distance $s(t)$ from the first flower at any instant is equal to a constant times the square of the time of flight. If the time to reach the second flower is also 2 seconds, what is the bee's velocity at any instant? Let's start by making a graph of distance traveled versus time. Making the graph is a straightforward algebra problem, so let's recall how to do it.

We can start by writing out the algebraic relationship between distance and time, namely,

$$s(t) = ct^2 \tag{1.2}$$

Furthermore, the distance traveled in 2 seconds is known (50 cm), so put in 2 seconds for $t$ and 50 cm for s and then solve for $c$. Did you get $12\frac{1}{2}$ for the answer?

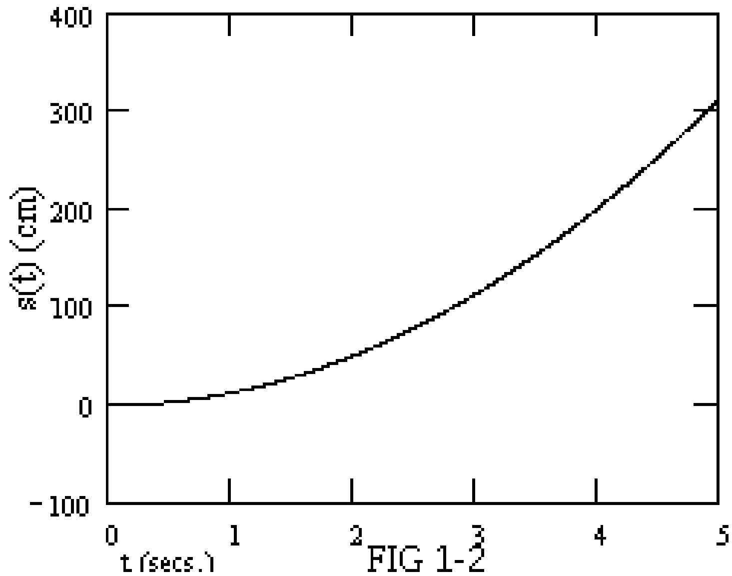



FIG 1-2

But the number $12\frac{1}{2}$ standing by itself is not enough. We have to remember that s was measured in cm and $t$ was measured in seconds. We do this by attaching "units" to our answer. The correct answer then, is $c = 12\frac{1}{2}$ cm/sec$^2$.

Now you are ready to plot your curve of $s$ versus $t$. Use values of $t$ at intervals of, say, 0.2 sec. to calculate 9 values of s (you *already* know where the bee is at time $t = 0.0$ and $t = 2.0$). Mark each point on a sheet of graph paper and then draw a smooth curve through the points (a flexible curve and a sharp pencil will help make your curve look neat). The result should look something like Fig. 1-2. Note that the vertical axis in Fig.1-2 is labeled $s(t)$. The parentheses mean, as you learned in algebra, that $s$ is a function of $t$. This notation, with parentheses, is one of the great puns of mathematics because sometimes the parentheses mean "multiply", as in $s$ times $t$, and sometimes they mean "function of". In this case, the "function of" case, they mean that $s$ and $t$ are related. For every value of $t$ there is a corresponding value of s, calculated from Eq. 1.2. The meaning of the parentheses in other instances will usually be clear from the context. Now use Fig.1-3 to decide if you want to go back and change your answers to Exercise 1.2.

The idea of relationships, like that between $s(t)$ and $t$, was not available to Zeno and Aristotle. It took about 1300 years after Aristotle's time for that idea to reach Western Europe, although the Hindus in India and the Arabs may have developed the idea a few hundred years earlier. That idea will now help us resolve Zeno's paradox, namely, how can we understand the idea of velocity at a particular instant if the velocity keeps changing?

We learned in problem 1.1 that the velocity of the bee's flight was given by the slope of the line showing the position of the bee as a function of time. So suppose we draw a straight line that touches the curve of Fig.I-2 at a single point, i.e., just grazes the curve, at $t = 1$

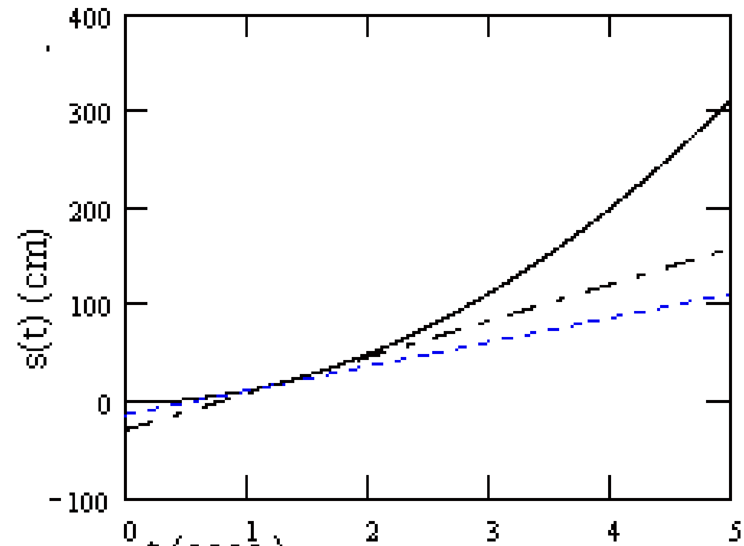


FIG 1-3

sec, as shown by the dashed line in Fig. 1-3. The entire dashed line represents an object (a bee, in this case) traveling with a *uniform* velocity of 12.5 cm/sec.

The dashed line touches the curve at a single point (at $t = 1$ sec). The dashed line is, then, *tangent* to the curved line (we will return to the subject of tangents). In other words, the dashed line and the curve are changing in the same direction at the point of tangency. The slope of the tangent line is just "the slope of the curved line at the single point, $t = 1$ sec".

The slopes will be different at different point of the curve, as is shown by the dash-dotted line in the figure. The dash-dotted line is tangent at the point $t = 1.5$ sec.

But the slope of a line on a distance-time graph, we have seen, corresponds to velocity. The dashed line therefore shows the velocity of the third bee at time $t = 1$ sec. If we measure the rise and run of the dashed line we will find that the velocity at $t = 1$ sec is just 25 cm/sec.

**Exercise**

**1.4** Use your graph to find the velocity of the third bee at $t = .5$ sec and $t = 1.5$ sec (You should get a value that lies between 30 and 40 cm/sec at $t = 1.5$ sec). (This problem relates to Exercise 2.12 later on.)

## 1.3 WHAT ABOUT TANGENTS TO CURVES?

We have constructed, in the previous section, a curve that describes the progress of an object, a bee, as time progresses. We also discovered that the *velocity* of the object, the *rate of change of its position*, at a particular instant is given by the slope of the tangent line to the curve at that instant. In a rough, poetic sense, the curve described the relationship between two quantities, and the slope of the tangent to the curve at any point of the curve gave the rate of change of that relationship at that point. We can carry this basic idea over to other kinds of interesting relationships than the relationship between the position of a moving object and time.

Chemists, for example, are interested in the relationship between the pressure and the temperature of a volume of gas. Biologists are interested in the relationship between the body temperatures and metabolic rates of various animals. Economists are interested in the relationships between the prices and the the demands for various commodities. These few examples should be enough to start you thinking of other kinds of relationships that can be discussed in the same way that we have discussed the flight of a bee between two flowers. The mathematician now asks, what do all these relationships have in common so that we can discuss all of them using the same mathematical tools?

The answer is really quite simple. We need to be able to describe the relationship with a curve, and we need to be able to construct a tangent to the curve at every point. Then the rate of change of the relationship at any point is given by the slope of the tangent to the curve at that point. We can understand this answer better by considering situations where a relationship cannot be described by a curve or where there is a curve but we cannot define the direction of the tangent at each point.

## Exercises

**1.5** I have a set of cards numbered from 1 to $n$. In how many different orders can I arrange the cards?

If $n = 1$ then there is only 1 possible way.

If $n = 2$ then there are 2 ways (1,2 or 2,1).

If $n = 3$ then there are 6 ways (write them out).

Can you now write out a formula that is valid for any $n$? Do so, and check it for $n = 4$.

Let $y(n)$ be the number of different orders in which $n$ cards can be arranged. Make a graph with $y(n)$ on the vertical axis, $n$ on the horizontal axis, and $n$ having values from 1 to 7 (hint: the vertical axis should range from 0 to 5040).

Now answer the following:

a. Is there a curve that describes the relationship between $y(n)$ and $n$?

b. What meaning does the curve have for values on the $x-$axis that are not integers?

c. Explain what shape the curve should have between integer values of n.

(If your answers to b. and c. are other than "I don't know", ask your instructor to place you in the advanced calculus course).

**1.6** Graph the following curve: for values of $x$ between 0 and 5 let $y = x^2$ ; for values of $x$ between 5 and 10 let $y = \frac{125}{x}$. You should be certain that the $y-$axis on your graph paper shows at least 25 divisions, and that the $x-$axis shows at least 50 divisions.

a. Can you draw a tangent to the curve at $x = 5$? Do so if you can.

b. Is there more than one possibility for the direction of the tangent at $x = 5$? If so, draw a second tangent.

Exercise 1.5 provides an example of a function that is only defined for integer values of its independent variable $n$. The function is only defined at a few points on the graph, so there is no "curve" associated with the function. When there is a curve, and that curve has no (vertical) jumps, we say the function is **continuous**. Exercise 1.6 gives an example of a function that is described by a curve, but there is a point on the curve where it is not possible to determine the direction of the tangent (at $x = 5$). We describe this situation by saying that because the curve comes to a point at $x = 5$ it is not **smooth** there.

We will be interested in curves that are continuous and smooth because those are the only kinds of curves that we will be able to work with in the first part of this text. Such curves always have tangents at every point, because having a tangent at every point is really our definition of "smoothness". One such curve is the circle. We will close this section by helping you to prove a proposition from geometry that was known to the ancient Greeks, namely:

### Theorem I
### A tangent to a circle is perpendicular to the radial line from the center of the circle to the point of tangency

(Note: this will be our first example of a "proof." We will prove this proposition by first stating the facts upon which we base the proposition. We then will assume that the proposition is false, and show that the assumption leads to a contradiction with the known facts. This method of proof is known, unsurprisingly, as proof by contradiction.)

We are given the following facts, including the definition of "tangent."

1. A tangent to a curve is a piece of a straight line that touches the curve at a single point but does not cross the curve.

2. All radii of a circle have the same length.

3. If a point does not lie on a straight line, then the perpendicular to the straight line from the point is the shortest line segment from the point to the line.

4. The whole is greater than any of its parts.

### Proof

Draw the circle with center at C and a line tangent to the circle at A. Draw the radius CA and let CBD be any other straight line from the center of the circle to the tangent line (see Fig. 1-4) intercepting the tangent line at D. (Fill in each blank where there is a question mark).

I. Assume that CA is not perpendicular to the tangent line, so that CA is not the shortest

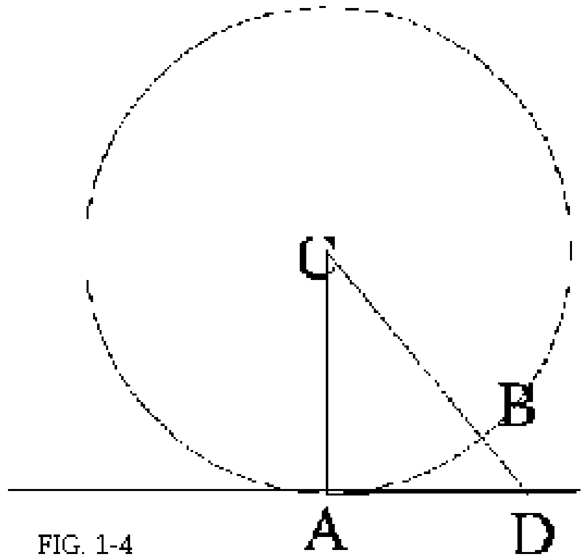



FIG. 1-4

line from C to the tangent line (according to fact # ......)

II. Let CD be the perpendicular from C to the tangent line at A, so that CD $<$ CA (using, again, fact #? ......)

III. But the tangent line touches the circle at only 1 point (fact # 1). Therefore point D must lie outside of the circle. So let B be the point where the CD intersects the circle.

IV. From III, CD $>$ CB, using fact # ......

V. But CA and CB are both line segments from the center of the circle to the circumference and are therefore both radii of the circle. Therefore, CA = CB (fact # ......)

VI. Since CD $>$ CB (from IV) and CB = CA, then CD $>$ CA (look ahead to Exercise 1.17).

VII. But CD $<$ CA from II, and CD $>$ CA from IV. Both cannot be true, so we are led to a contradiction if CA is not perpendicular to the tangent line.

### Exercise

**1.7** Suppose that the curve is an ellipse instead of a circle. Which fact would no longer be relevant? Does this mean that a line from the center of an ellipse to the point of tangency of a tangent line need not be perpendicular to the tangent line?

### Additional Exercises for Chapter I.

**1.8** We can define a **discontinuity** in a curve as a place where the curve is not continuous, i.e., where it has a vertical jump. Explain why a position vs. time curve for **you**, of the kind shown in Fig.I-2, *can never have a discontinuity*, i.e., can never be **discontinuous**.

**1.9** A bee flies with a velocity of 8 m/sec for 14 seconds. How many meters did the bee travel?

**1.10** The bee in Exercise 1.**1.9** traveled $d$ meters at velocity $V$ and $e$ meters at velocity $U$. Write an algebraic expression for the total time that the it took the bee to cover the $d+e$ meters. The total distance traveled divided by the total time elapsed during the travel is called the *average velocity* during the journey. Write an expression for the average velocity $v_{av}$ during the flight.

**1.11** In Exercise 1.10, suppose $d = 9$, $e = 12$, $V = 6$ m/sec and $U = 4$ m/sec. Find the average velocity $v_{AV}$ of the bee during the its 21 meter flight. What constant velocity would be required for a bee to make the same flight in the same time?

**1.12** Suppose many bees make the same flight with the same velocity $V$(first 9 meters) but varying velocities $U$ (last 12 meters). Write an expression for the average velocity of each bee (which will, of course, depend on $U$) and call it $v_{AV}(U)$, the average velocity of each bee as a function of $U$.

**1.13** Take a sheet of graph paper with at least 10 large divisions along each axis and take the origin at the center of the sheet. Number the large divisions in each direction so that both the $x-$ and $y-$ axes include the ranges $-5$ to $+5$. Use a compass to draw a circle with a radius of 3 divisions centered at the origin. The circle will pass through the point (2.6, 1.5), to an accuracy of 1 decimal place. Use a straight-edge to draw the tangent to the circle at this point. Find the value of slope of the tangent line by measuring the rise and the run of the tangent line (the longer you make the line, and the rise and the run, the more precise your result will be). Be sure to note whether the value of the slope is a positive or negative number.

**1.14** Here are some algebra relations that we will need later, so this is a good time for you to work them out:

$$\frac{x^2-h^2}{x-h} = ? \qquad \frac{x^3-h^3}{x-h} = ?$$
$$\frac{x^4-h^4}{x-h} = ? \qquad \frac{x^n-h^n}{x-h} = ?,$$

$n$ an integer.

**1.15** Solve for $x$ (be sure to find both solutions)

$$\frac{4-x}{3+x} = \frac{4-5x}{3+2x}$$

**1.16** A few more algebraic relationships:

$$\frac{(x+h)^2-x^2}{h} = ? \qquad \frac{(x+h)^3-x^3}{h} = ?$$
$$\frac{(x+h)^4-x^4}{h} = ? \qquad \text{The first term of } \frac{(x+h)^n-x^n}{h} = ?,$$

$n$ an integer. (The first term is the one with the highest power of $x$)

**1.17** In the proof that the tangent to a circle is perpendicular to the radius at the point of tangency, what additional facts do we need to justify step VI? Is it always permissible to substitute a quantity for its equal in an algebraic relationship (such as $>, <$, or $=$)?

**1.18** USE YOUR CALCULATOR to calculate $739^{929}$ to six digit accuracy (the answer should be a number between 1 and 10, expressed to six digits, times a power of 10). (SHOW ALL WORK). We include this exercise to discourage the use of calculators, which, in any event, are untrustworthy. We will advise you of circumstances where calculators may be used.

## 1.4 ...AND, IN CLOSING

When we discussed, in connection with Fig. 1-2, the notion of distance as a function of time, described by the notation $s(t)$, we treated time $t$ as a variable. The graph of $s(t)$ versus $t$ is just a curved line on a piece of paper. It is then easy for us to speak of "$s$ at a particular value of $t$" because that is just one of the points on the curved line.

The idea of treating time as a mathematical quantity, like distance, was a gigantic intellectual achievement. It occurred for the first time in all of human history, as far as we know, in the writings of a man named Galileo Galilei, in Italy during the 1600's. The ancient Greeks would no doubt have been horrified by Galileo's ideas because they had an almost mystical concept of time. As one of them, a man named Chrysippus, once wrote: "part of the present is in the past and part in the future." In other words, it is not possible to define a single instant of time, as Galileo did, as being the "present".

But, you might say, we are all beyond this. Why bother ourselves with ancient Greek ideas? The answer is that many of these ideas are very close to the viewpoints of many students who are just beginning to learn mathematics and science. The fact that it took a million or so years of human history to get to Galileo's ideas warns us that these ideas are extremely far from obvious.

The first idea of this book, which we invite you to struggle with, is that rate of change of a curve at a point is measured by the slope of the tangent to the curve at the point.

### PUZZLE CORNER

NO CREDIT
BUT TRY IT ANYHOW
(answer at end of next chapter)

You are in a land where everyone is either a liar or a truth teller. You desperately need a truth teller for a business deal. You have lunch with 3 people, A, B, and C and ask if any of them is a truth teller.

They answer as follows:
A: "There are 3 truth tellers here".

B: “No, only 1 of us is a truth teller.”
C: “The second person is telling the truth.”
Which, if any, are the truth tellers?

# Chapter 2

# FINDING SLOPES, PART 1

The point of the first chapter was this: when a curve describes the relationship between two quantities, then the slope of the tangent to the curve at any point of the curve gives the rate of change of that relationship.

Our task in this chapter is to learn how to calculate the slopes of the tangents to some simple curves.

## 2.1 THREE EASY PIECES: THE CONSTANT, THE STRAIGHT LINE, AND THE CIRCLE

### 2.1.1 THE CONSTANT

We've been discussing functions, or relationships, involving time. Distance traveled by a moving object, for example, was expressed as $s(t)$, meaning that the distance changed as $t$ changed. But, as pointed out in the last chapter, there are other important relationships that don't necessarily involve time. So just to keep the discussion general we'll talk about relationships involving something called $y(x)$, that is, a quantity $y$ that changes as the quantity $x$ changes. You may, if you like, think of $y$ as something like distance and $x$ as something like time. What is important is that $y$ changes as $x$ changes.

The curves that we are discussing are the curves that show the relationship between $x$ and $y$. We show these curves on graphs, just as you learned to do in your algebra courses, with $x$ on the horizontal axis and $y$ on the vertical axis. The first curve that we will deal with shows the situation when the value of $y$ is the same for every value of $x$. That is

$$y(x) = c \tag{2.1}$$

where $c$ is some number. Fig. 2-1 shows three constant functions corresponding to $c$ equal to 1.5 (solid), 1 (dash-dot) and -7 (dashes).

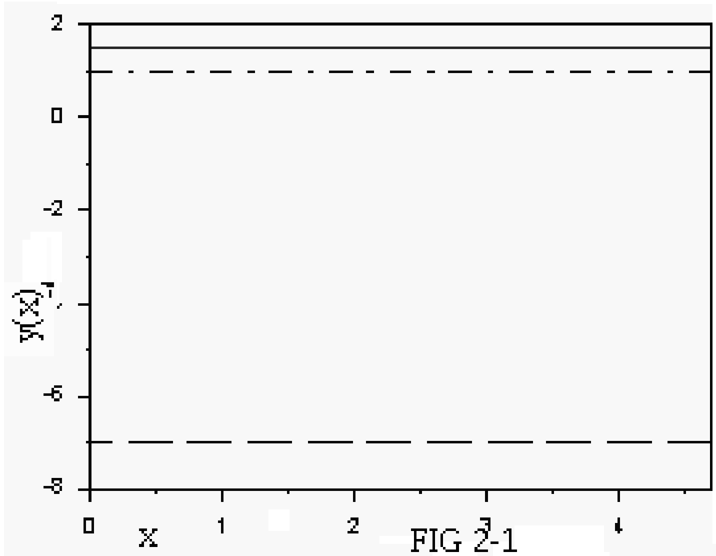



FIG 2-1

The lines are obviously parallel, so they must all have the same slope. The **slope**, as we remembered in Chapter I, is the **rise** over the **run**. The rise is the difference in $y$ at two different values of $x$. The difference in the $x$ values is the run. But the constant functions have the same value of $y$ at every value of $x$, so the value of the rise is zero for every value of the run. Zero divided by any number is zero, so the constant functions have zero slope.

What could have been more obvious? The slope of a straight line is the (trigonometric) tangent of the angle between the line and the $x-$axis. Constant functions are described by lines parallel to the $x-$axis, so the angle they make with the $x-$axis is zero. The tangent of the angle zero is zero, so the value of the slope of a constant (FUNCTION) is equal to zero.

**EXERCISE**

**2.1** Write out two explanations why the number zero divided by any (non-zero) number is zero. The first explanation might be geometrical, that is, if a line has zero width how wide can a fraction of that line be? For the second explanation, you might consider that dividing by a number is the same as multiplying by 1 divided by that number. What do you know about zero multiplied by anything?

### 2.1.2 THE STRAIGHT LINE

You learned in algebra that the equation of any (non-vertical) straight line on an $x - y$ graph can be written

$$y(x) = ax + b \tag{2.2}$$

Fig 2-2 shows a graph of the straight line that results when $a = 4$ and $b = -9$.

Take the run between any two points on the curve and measure the rise between the same two points. The value of the rise divided by the run then gives the slope. Did you get a slope of 4 for the curve shown in the Figure?

Suppose that $x_1$ and $x_2$ are the $x-$ values at the two points. The $y-$values are then $y(x_1)$ and $y(x_2)$, so that the rise is
$y(x_1) - y(x_2)$. But

$$y(x_2) - y(x_1) = (ax_2 + b) - (ax_1 + b) = a(x_2 - x_1). \tag{2.3}$$

We get the slope by dividing the rise by the run, so that

$$\frac{y(x_2) - y(x_1)}{x_2 - x_1} = a \tag{2.4}$$

Equation 2.4 reminds us of something else that we learned in elementary algebra. In the equation for a straight line on an $x-$y graph,

$$y(x) = ax + b \tag{2.5}$$

the constant a is the value of the slope of the line.

## EXERCISES

**2.2** Finish the sentence: The constant b in Eq. 2.2 represents the value of . . . .

**2.3** What is there about the right side of Eq. 2.4 that tells us that it didn't matter which two points on a straight line we used to determine the slope?

**2.4** We often describe parallel lines as lines that can never intersect, no matter how far they are extended. Are all of the straight lines that are described by Eq. 2.5 with the same value of a, but different values of b parallel to each other? (Hint: Think about the difference of two such equations.) Explain your answer. The next more complicated kind of equation will occur in Ex. 2.13.

**2.5** Temperature can be measured either on a Fahrenheit or Centigrade scale. Let $x$ stand for the Fahrenheit reading, and let $y$ stand for the Centigrade reading. The relationship between the two scales can be described by a straight line. Find the equation of the line from the following facts:

(a) Pure water at sea level boils at $100°$ Centigrade and $212°$ Fahrenheit;

(b) Pure water at sea level freezes at $0°$ Centigrade and $32°$ Fahrenheit. (Hint: solve this problem by solving two equations in two unknowns. Check your answer by showing that $y$ and $x$ are equal at a temperature between $-35°$ and $-50°$, and find the temperature at which they are equal.)

**2.6** What is the slope of the Centigrade versus Fahrenheit line in Exercise 2.5? Explain why the slope has units of $°C/°F$ (C and F stand for Centigrade and Fahrenheit, respectively). Determine from the value of the slope which is larger, $1°C$ or $1°F$?

### 2.1.3 THE CIRCLE

The slope at a point on a circle is also an "easy piece" because we know that a tangent to a circle is perpendicular to the radial line from the center of the circle to the point of tangency. This was Theorem I in Chapter I. But if we know an equation that describes the circle, then we know the slope of the radial line; it is just the rise over the run from the center of the circle to the point of tangency. So our problem is to find the slope of a line that is perpendicular to a line of known slope on a graph.

Notice what we set out to solve a very specific problem: finding an expression for the slope of a tangent to a circle. We discover that in order to solve that problem, we can solve a much more general problem: given a line of known slope, find an expression for a line perpendicular to the given line. When we solve the more general problem, we can put that solution into our notebooks for later use. Our notebooks then become our toolboxes for solving future specific problems. Making toolboxes is what learning mathematics is all about.

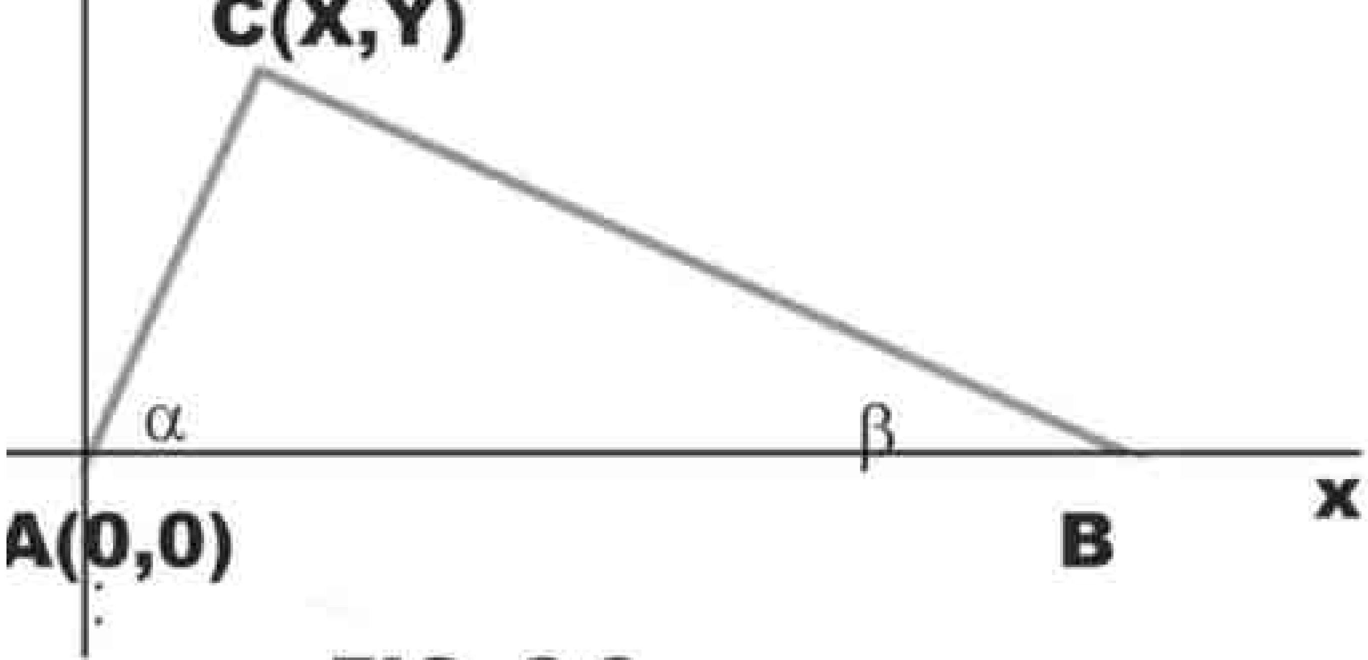



FIG. 2-3

You already have some tools in your toolboxes, namely, everything that you remember from algebra, trigonometry, and geometry.

We first answer the general problem, and then explain how we arrived at the answer. The "how" is the important part of what we want to teach.

We pompously state the answer as

**Theorem 2: Given a line of slope $m$, any perpendicular line has slope $-\frac{1}{m}$.**

The "how" is what we call the "proof" of the Theorem 2. Note that it is not enough to show that Theorem 2 is true for some particular value of m, because it might be false for other particular values - in which case Theorem 2 would not be a theorem. But it clearly doesn't matter if we make the given line go through the origin - after all, it is parallel to a line going through the origin and parallel lines have the same slope.

**Proof**

What we have to work with is a given line having slope m on a graph and a line perpendicular to the given line.

I. Let line AC in Fig. 2-3 be the given line, and let point A be at the origin of the graph and point C have the coordinates $x = X$ and $y = Y$. $X$ and $Y$ can be any 2 numbers whatsoever.

II. AC is then a line with a slope on a graph. The slope is the rise $Y$ over the run $X$, so we have the slope m given by

$$m = \frac{Y}{X}. \tag{2.6}$$

It is important to notice that m can have any numerical value, depending on how we choose to draw the line AC.

III. Construct a line BC that is perpendicular to the given line at C and crosses the $x$-axis at some point B. This construction gives us a line that is perpendicular to a given line with known slope.

Next we must ask ourselves, what are the properties of this combination of lines that permit us to obtain Theorem 2 as a result (there are usually many different ways to prove any theorem). One possibility is to notice that ABC is a right triangle, so maybe a little trigonometry can help.

IV. Label the angle at A $\alpha$ and the angle at B $\beta$, as in Fig. 2-3.. The slope m of AC is just $\tan\alpha$, which is not only $\frac{Y}{X}$ but is also side opposite over side adjacent to angle $\alpha$ in the triangle ABC. That is,

$$m = \frac{Y}{X} = \frac{\overline{BC}}{\overline{AC}} \tag{2.7}$$

V. The slope of the line BC, which is what we are after, then must have magnitude $\tan\beta$, or $\frac{\overline{AC}}{\overline{BC}}$, which is just $\frac{1}{m}$ from Eq. 2.7. VI. Now we are finished, if we are careful about the sign of the result. The line BC has a positive rise with a negative run as we go from point B to point C. The slope is therefore negative, so we have the final result

$$\text{slope of perpendicular to AC} = -\frac{1}{m} \tag{2.8}$$

*Q.E.D.* (that which was to be proven)

We proved the Theorem 2, then, by noticing that trigonometry gave us a relationship between the slope of a line and its perpendicular. The clever step in the proof was to put in the line AB to make a right triangle, so we could see that trigonometry could be used. Most proofs involve coming up with a "clever step".

With Theorem 2 in our toolbox, we can go back to talking about a circle. The task is to determine the slope of a radial line to a point on a circle; we will then know immediately the slope of a tangent to the circle at that point.

We will keep things simple by supposing that the center of a circle of radius $r$ is at the origin $(0,0)$, as in Fig. 2-4. Later we will learn how to move the center to any other point. If the coordinates of any point on the circle are $(x, y)$, then the Pythagorean theorem tells us that

$$x^2 + y^2 = r^2. \tag{2.9}$$

We unfortunately can't write a single equation for $y(x)$ to describe the entire circle. We can, however, write

$$y(x) = +\sqrt{r^2 - x^2} \tag{2.10}$$

for the part of the circle above the $x-$axis, and

$$y(x) = -\sqrt{r^2 - x^2} \tag{2.11}$$

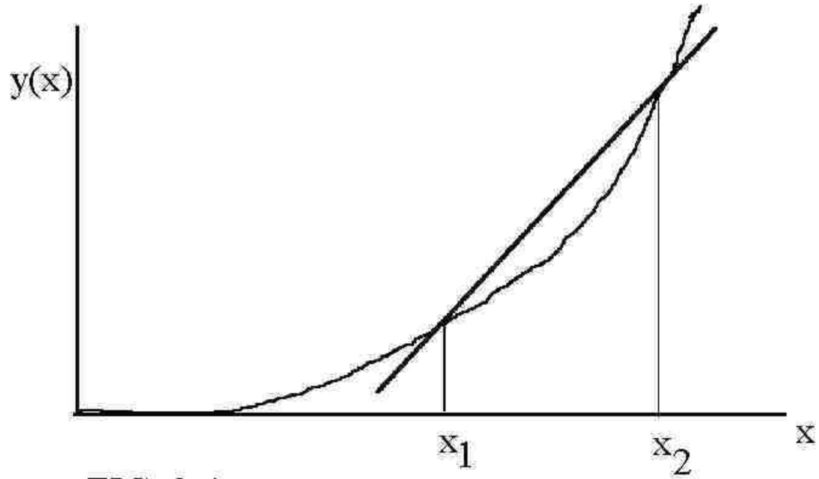



FIG. 2-4

for the part of the circle below the $x-$axis.

Suppose that point C in Fig 2-4 has coordinates $x = X$ and $y = Y$, so that the radial line is just like the line AC (=0C) in Fig. 2-4. The slope of the radial line in the figure is just (rise over run) $Y/X$, namely, the trigonometric tangent of the angle between the radial line and the $x-$axis. The slope of the tangent line is then given to us by Theorem 2, namely, $-X/Y$.

The result of this section is: **The slope of the tangent, at a point $(X, Y)$ on the circumference, of a circle centered at the origin, is $-\frac{X}{Y}$.**

## EXERCISES

**2.7** Suppose that $y(x) = 3x + 5$ is the equation of a given line. Give the equation of $y_p(x)$, all the lines perpendicular to the given line. Hint: "all the lines" means that your equation will have one undetermined constant. Call this constant "$b$".

**2.8** In Exercise 2.7, determine the value of $b$ so that the perpendicular line intersects the given line at the point $x = 4$. Hint: What must be true of the values of $y(x)$ and $y_p(x)$ at the point of intersection? Show that your answer is correct by making a careful graph of $y(x)$ and $y_p(x)$. Use millimeter graph paper and a ruler.

**2.9** (a). We remarked in proving Theorem II that one cannot prove a general proposition by giving examples where the proposition is true. Describe what is wrong with the following "proof" of a "Theorem": Theorem: All integers are odd numbers. Proof: The numbers 1, 3, and 7 are integers. They are all odd numbers. Conclusion: All integers are therefore odd.

(b). Can you disprove the theorem by giving one example where the theorem fails? Such examples are called "counterexamples".

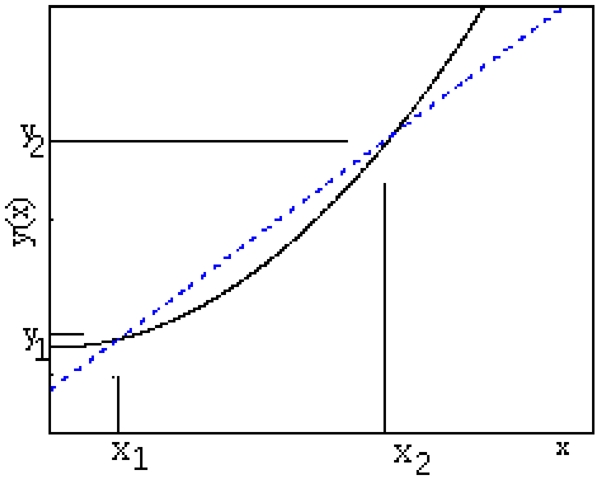



**FIG. 2-5**

## 2.2 THE QUADRATIC TEACHES A GENERAL METHOD FOR SLOPES

The paradox that so troubled Zeno shows itself if we try to use the left-hand side of Eq. 2.4 to find the slope at a point, call it $x_1$, of the quadratic equation $y(x) = cx^2$ where $c$ is some constant. Then

$$\begin{aligned} \text{slope} &= \frac{y(x_2) - y(x_1)}{x_2 - x_1} = \frac{cx_2^2 - cx_1^2}{x_2 - x_1} \\ &= c(x_2 + x_1) \end{aligned} \tag{2.12}$$

We wanted to find the slope of the quadratic at a point whose $x-$ coordinate, say, is equal to $x_1$. But, as Zeno recognized, the process of taking the slope involves two points because the slope is the rise of $y$ between two points divided by the run along $x$ between the same two points. So we have a result that depends, not only on the point with $x = x_1$, where we want the slope, but also on some other point with $x = x_2$.

The problem should become clearer if we sketch the situation, as in Fig. 2-5.

We see in the figure that the straight line, which we want to be our tangent line, cuts the curve at two points, with $x-$values $x_1$ and $x_2$. But we only want the tangent line to touch the curve at the one point, namely where $x = x_1$. What to do? Choose $x_2$=$x_1$! Then Eq. 2.12 becomes:

$$\text{slope} = 2cx_1 \tag{2.13}$$

(Compare this result with the equation in Ex 2.3 and your answer to that exercise.)

You can easily check to see that this answer is correct by doing the next exercise.

## EXERCISE

**2.10** The equation of the curve in Fig. 2-5 is $y(x) = cx^2$. The equation of the straight line that is tangent at $x = x_1$, which we'll call $y_{tangent\ at\ x_1}(x)$ must be given by $y_{tangent\ at\ x_1}(x) = (2cx_1)x + b$, where $b$ is a constant to be determined. Determine $b$ from the condition that $y(x)$ and $y_{tangent\ at\ x_1}(x)$ must meet at the point with $x = x_1$. Explain (in writing - this will take some thought) why the fact that the equation $y(x) - y_{tangent\ at\ x_1}(x) = 0$ has only a single solution (when your value of $b$ is used) means that $y_{tangent\ at\ x_1}(x)$ is probably the tangent line. Must it be?

Eq. 2.12 suggests that there is a general method for finding the slope of the tangent to any function, $y(x)$, at some point $x_1$. Just find the result of the division, $\frac{y(x_2)-y(x_1)}{x_2-x_1}$, and then let $x_2$ be equal to $x_1$. We will explore this idea further in the next chapter.

## EXERCISES

**2.11** (a) Explain why, in the expression, $\frac{y(x_2)-y(x_1)}{x_2-x_1}$ in Eq. 2.12, it is necessary to do the division first, and only after that letting $x_2$ be equal to $x_1$.

(b) Illustrate the point that you made in part (a) by using $y(x) = cx^3$ as an example.

**2.12** We found that in the equation for a straight line, $y(x) =$ ax + b, the constant a is the slope. Use this fact to argue that the slope of the function $y(x) =$ a constant must always be zero.

**2.13** Use the result of this chapter to check your answers to problem I.4. Show all of your work.

**2.14** Suppose that the distance traveled by an automobile as a function of time is given by the formula:

$$s(t) = pt^2 + qt + r \tag{2.14}$$

where $p, q$ and $r$ are constants. Use the principles that you learned in this chapter to find an expression for the automobile's velocity, call it $v(t_1)$ at the instant of time $t_1$.

**2.15** Make a graph of the distance function in Eq. 2.14 for the case where $p = 5\text{m/sec}^2, q = 2\text{m/sec}$,r= 4m, and $t$ ranges from 0 to 5 seconds. Check your answer to Exercise 2.14 by drawing in the tangent line that shows the slope at $t_1 = 2$ sec.

**2.16** It is known that the distance traveled by a falling object obeys Eq. 2.14, where s(t) is the distance fallen at any time and $t$ is the elapsed time.

(a) How many values of s(t) would you need to determine the values of p, q, and r?

(b) You are given that $r = 0$ and that $s = 21$ ft at 1 sec and 159 ft at 3 sec. Can you now find the values of $p$ and $q$? Do so if you can.

(c) Find the time or times at which s(t) is zero.

(d) Check your work by making a graph, as in Ex. 2.15. You may use a graphing calculator for this step.

(e) Find the value of the slope of the tangent to the curve at each point where s(t) is zero. Do this by using the equation for slope that you obtained in Ex. 2.12.

## 2.3 ...AND, IN CLOSING

If $y(x)$ is some function of $x$ that is "smooth", then the graph of $y$ versus $x$ describes some sort of curve. We learned in this chapter that the slope of the tangent at any point on a quadratic curve when $x$ has some value $x_1$ can be found by doing the division $\frac{y(x_2)-y(x_1)}{x_2-x_1}$ (we call this the "ratio of the change in $y$ to the change in x") and then letting $x_2 = x_1$. If we let $x_2 = x_1$ *before* taking the ratio, the result will just be 0/0, which is a meaningless expression (we called it "indeterminate" in algebra courses). So by doing the division first and then letting $x_2 = x_1$, we have found a way of giving a definite meaning to an expression which was indeterminate and appeared to be meaningless. In the next chapter we will extend this same process to other curves than quadratics.

It should be obvious that for almost any function $y(x)$ the quantity $\frac{y(x_2)-y(x_1)}{x_2-x_1}$ is the slope of a line on an $x - y$ graph because it is just the rise divided by the run between two points on the graph (or "the ratio of the rise to the run"). We will study, in the next chapter, how to make this ratio meaningful when both the rise and the run are zero! We will do this by introducing a new concept—the idea of quantities that are so small that they are "almost zero."

### PUZZLE CORNER NO CREDIT BUT TRY IT ANYHOW

A bricklayer has 8 bricks. Seven of the bricks weigh the same amount, but one is a little heavier than the others. The bricklayer has a balance. How can she find the heavy brick in only two weighings.

### ANSWER TO THE CHAPTER I PUZZLE

All are liars:

(a) If A's statement is true, then B's denial is false, and B is a liar. But if B is a liar, then A's statement is false. Therefore A's statement cannot be true and A is a liar.

(b) If B's statement is true then either B or C must be the truth teller, since we know that A is not. But if C is the truth teller, then B cannot be, making B's statement false. Thus, either B is the only truth teller, or B is a liar.

(c) If B is a truth teller, then C's statement is true. But If B is a truth teller, then B must be the ONLY truth teller, so C's statement must be false. Thus C is a liar, and C's statement must be a lie, making B's statement false.

# Part II

# CONCLUDING REMARKS

# Chapter 30

# WHERE HAVE WE BEEN, WHERE SHOULD WE GO NEXT?

## 30.1 WHERE HAVE WE BEEN?

We began this book with the question, how does one describe motion? This question led us to the study of tangents to curves and, eventually to the study of curves as such. We called the process of finding a tangent to a curve at every point, “finding the slope of a curve”.

The slope of a curve, at every point of the curve, can be plotted on a graph to make a new curve that we can, somewhat sloppily, call the “derivative curve.” This fact permits us to ask the question of any curve, “of what curve is this curve the derivative?” We called the process of answering this question, “finding the anti-derivative” of a curve. Anti-derivatives, we are told, are often called “integrals”. More importantly, we learned that the integral of a curve describes (or measures) the “area under the curve”.

The two preceding paragraphs summarize the two fundamental ideas that make up calculus. Everything else is machinery - machinery for describing curves, for finding the derivatives and anti-derivatives of given curves, and for guessing the shapes of curves.

**Exercise**

30.1 Sketch the curve of the function $y(x) = bx^4 - ax^3$, where a and b are both positive constants that do not have any specific values.

(a) At how many points, if any, does the curve cross the x-axis?

(b) What does the curve look like when x is very close to zero (which term dominates?)?

(c) What does the curve look like when x is very large in magnitude (which term dominates?)?

(d) How many stationary points, if any, does the curve have, and are these points maxima, minima, or points of inflection?

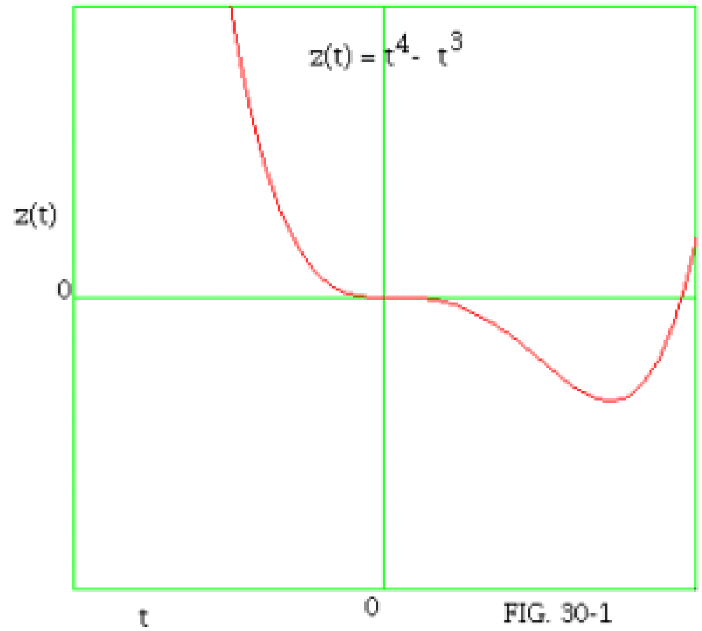


FIG. 30-1

(e) Now sketch the curve.

Fig. 30-1 shows one such curve, namely $z(t) = t^4 - t^3$. Must all curves of Exercise 30.1 (for all positive values of a and b) resemble Fig. 30-1? (Hint: What happens if you define a new variable $t$ by the relation $x = \frac{a}{b}t$?). We also began to explore some of the "real life" applications of these very mathematical (that is, having to do with algebra and geometry) notions. If, for example, a curve shows the position of an object along line as a function of time, then the derivative curve shows the velocity of that object along that line as a function of time. That is the example that we started this book with.

If, on the other hand, a curve shows the velocity of an object as a function of time, then the integral curve shows the position as a function of time. But we can look to other examples than just position and velocity. If, for example, a curve describes the rate of increase of an investment as a function of time, then the integral curve describes the value of the investment at any time. Or, to take an example from art (meaning geometry), if a curve shows the shape of the boundary of an area, then the integral curve shows the amount of area within the boundary.

All of these examples have one feature in common. Each example involves a variable that depends upon ("is a function of") one other variable. That fact shows us the crucial limitation of the material that we've studied so far, because real life "functions" usually depend upon many variables. Objects generally move, not upon a line, but in a space with three dimensions. Areas bounded by curves are interesting to artists, but architects (who are also artists) are more interested in volumes that are bounded by surfaces. And, for a final example, economists are usually interested in market behavior that depends upon many variables such as tax rates, consumer confidence, dividend policies and many other factors. So we must expand our vision, in order to make calculus useful in practical situations, to functions that depend upon many variables.

## 30.2 WHERE SHOULD WE GO NEXT?

### Exercise

30.2 Here is a very simple example from economics of a two variable problem. Suppose a manufacturer manufactures two products, call them X and Y. Let $x$ be the income from product X and $y$ the income from product Y. Also, let the cost associated with $x$ dollars of income from X, with no Y produced, be $3x^2$. Similarly, let the cost associated with $y$ dollars of income from Y, with no X produced, be $4y^3$. Finally, let the cost of respective incomes $x$ and $y$ when X and Y are produced together be $2xy$.

The function that expresses the manufacturer's net income, which we might refer to as "the revenue function R(x,y)", is then

$$R(x, y) = x + y - 3x^2 - 4y^3 - 2xy \tag{30.1}$$

The question that the manufacturer wants answered is, how much of X and how much of Y should I produce to maximize my revenue? Let's work out the answer. But first, let's inspect a picture of the revenue function (see Fig. 30-2).

(a). Find the derivative of R with respect to $x$, treating $y$ as a constant and set that derivative to zero. Then find the derivative of R with respect to $y$, treating $x$ as a constant and set that derivative to zero. Now you have two equations in two unknowns.

(b) Solve the equations and express your answers as irrational fractions, remembering that both $x$ and $y$ must be positive quantities. Now check your answers by noting that the approximate answers are:

$$x = .078, \quad y = .27 \tag{30.2}$$

We chose the "cost constants" 2,3 and 4 so that the problem would have a solution. Can you find a different set of constants so that there is no solution (perhaps because the square root of a negative number is involved)? Also, can you show that your solution to Exercise 30.2 is a maximum, and not a minimum or other kind of stationary point. If you can, you have a great start on your next calculus course.

The subject of differential equations, even for functions of a single variable, is one that can also be greatly expanded. We have dealt in this book with a very simple differential equation, namely,

$$\frac{dy}{dx} = ky \tag{30.3}$$

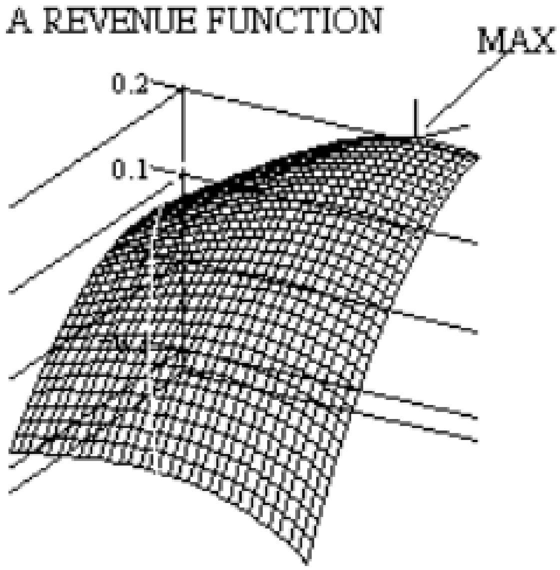



FIG. 30-2

and learned that the solution to the equation is the exponential, $y = ce^{kx}$, where c is some (arbitrary) constant. Note that the right hand side of Eq. 30.3 involves only the first power of y, multiplied by a constant, and the left hand side involves only the first derivative. The solution in this case was an elementary function.

**Exercise**

30.2 A much more complicated looking equation that turns out to have a relatively simple solution, is

$$\frac{d^2y}{dx^2} + (\frac{2}{x})\frac{dy}{dx} + y = 0 \tag{30.4}$$

(a) Solve Eq. 30.4 by making the educated guess that $y(x) = \frac{z(x)}{x}$ where $z(x)$ is a new unknown function. (You will learn later how to make "educated" guesses). Now find a new equation for z(x) and find that you can immediately write down a solution involving trigonometric functions.

(b) Note that Eq. 30.4 involves a second derivative. This means that the solution has two arbitrary constants. That is, there are two different trigonometric functions that solve the equation for z(x). Make sure that your solution has two arbitrary constants.

Learning how to solve differential equations is really the process of learning how to make educated guesses.

We did not motivate the differential equation in Exercise 30.2 (which is an example of an equation called Bessel's equation). We make up for that omission by turning to an example that arises in the study of the art of war.

### Exercise

30.3 The British engineer, F.W. Lanchester, studied, during World War I, the importance of an army having numerical superiority in modern combat. In ancient hand- to-hand combat a numerically superior army could only bring a small number of its soldiers at any given time against a small opposing force. Modern weaponry, on the other hand, permits each member of the one force to bring firepower to bear upon all of the other force.

Lanchester accordingly made the following mathematical model of a modern combat situation. Let $x(t)$ and $y(t)$ be the respective sizes of the two armies at any time t after a battle has begun, and let $x_0 = x(0)$ and $y_0 = y(0)$ be the initial sizes of the two armies. Lanchester then assumed that the loss rate in army $x$ at any instant is proportional to the number of soldiers in army $y$, and vice-versa. That is,

$$\frac{dx}{dt} = -ay, \quad \frac{dy}{dt} = -bx \tag{30.5}$$

where $a$ and $b$ are positive constants. The values of $a$ and $b$ represent the respective firepower effectiveness of the $y$ and $x$ armies, large values corresponding to great effectiveness.

(a) Find, from Eq.30.5 an expression for $\frac{dy}{dx}$ that does not involve the time, and then integrate the expression to obtain a relationship between $x$ and $y$ that involves the constants $a$ and $b$. Sketch the resulting relationship for the case where $a$ is bigger than $b$ (without using specific values for $a$ and $b$). (Hint: if you have an undetermined constant, you can determine it in terms of the conditions at the beginning of the battle).

(b) Suppose $a$ and $b$ have the respective values 0.4 and 0.1, $x_0 = 600$ and $y_0 = 500$ so the relationship is as shown if Fig. 30- 3. How many soldiers in the $y$ army have been lost when there is nobody left in the $x$ army? Show the calculation on which you base your answer.

(c) Suppose that $y_0^2$ is exactly equal to $\frac{ax_0^2}{b}$. What can you say about who will win the battle? (Hint: what will be the value of x when there is nobody left in the y army?).

(d) Differentiate both sides of the first Eq.30.5 with respect to the time. Can you now obtain a differential equation for $x$ that does not involve $y$? Would that be true if the right hand sides of Eq.30.5 involved higher powers of $x$ and $y$ than the first power? Might there be a benefit in having a model that is easily solved, even though it might not be as realistic as a more complicated model?

The example in the last exercise is intended to demonstrate the principle that a small force with superior firepower can destroy a much larger force while taking on relatively few casualties. Lanchester's very simple model, while it might not be fully realistic, provides a demonstration of this principle.

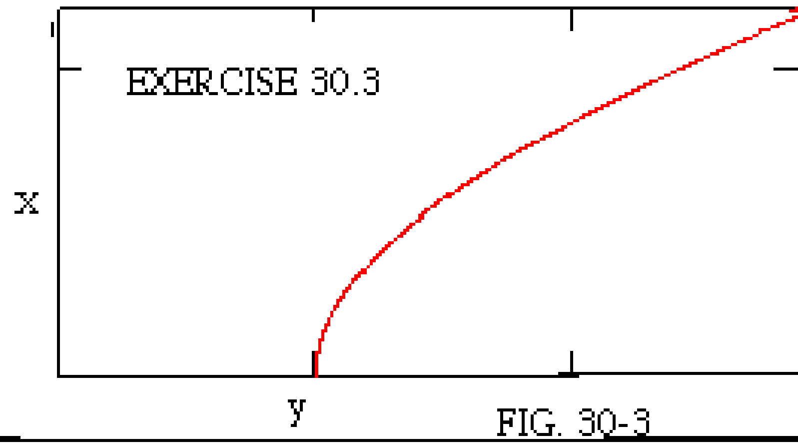


FIG. 30-3

## 30.3 ...AND, IN CLOSING

In the oversimplified mythology that teachers often pass on to their students, calculus emerged full-blown from the brains of Sir Isaac Newton (1642-1727) and Gottfried Wilhelm Leibniz (1646-1716). Newton invented the calculus as a tool to explain the motion of the planets around the sun. Leibniz happened to develop a large part of the subject concurrently and independently, apparently as an intellectual exercise. Thereafter Leibniz and Newton engaged in an angry correspondence over the question of who should get credit for the discovery.

The truth is much more complex and has been engagingly discussed by the mathematical physicist V. I. Arnold in his book *Huygens & Barrow, Newton & Hooke* (Birkhäuser, Boston, 1990). Barrow was a professor at Trinity College, Cambridge, and lectured on mathematics there. Newton attended Barrow's lectures, and Leibniz bought one of Barrow's books in 1673, so scholars now argue over whether Leibniz's invention of the calculus was really all that independent of Newton's.

What was in Barrow's lectures that scholars think may have been the germ that suggested the calculus to both Newton and Leibniz? Barrow developed the principle that, in Arnold's words (p. 41 of Arnold's book), "there is a duality between problems about tangents and problems about areas." "Duality" means that if, on the one hand, you solve a problem about the tangents to a curve, then you learn something about the area under the curve. On the other hand, if you solve the problem of finding the area under a curve, then you have learned how to construct tangents to the curve.

Barrow's duality principle is known today as the Newton-Leibniz formula which relates the integral of the derivative of a function to the function itself.

**Exercise**

30.4 Find the Newton-Leibniz formula in the text and state in words which part of the formula represents the area under a curve and which part represents the tangents to the same curve.

Barrow's duality principle says, in modern language, that in either case we are looking for a function. Once we have found the function, we know how to find its tangents (the derivative function) and the area under the curve that describes the function (the integral function).

Barrow's duality principle is what this book is about.

## PUZZLE CORNER
## NO CREDIT BUT TRY IT ANYHOW

Four cockroaches are at the four corners of a square with 1 foot sides. When the light goes off, each cockroach proceeds directly toward the moving cockroach on his left, each cockroach moving at the same velocity. When the cockroaches meet at the center of the square, how far has each cockroach traveled. (The answer in *Intriguing Mathematical Problems* by Jacoby and Benson (Dover 1996).

## ANSWER TO THE CHAPTER 29 PUZZLE

C knows she is wearing a white hat by elimination: When the blindfolds were removed, everybody knocked once, and C could see two white hats. Suppose that C were wearing a black hat. Then A sees a white hat on B and knows that B must see a white hat on A. So that A knows that A must have a white hat. But A did not knock twice, so A does not know the color of his hat. Similarly for B. Therefore A and B each see two white hats. Thus, C is the first to realize that all the hats are white and knocks twice.