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CONCEPTUAL CALCULUS

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January 18, 2000

Chapter 3

FINDING SLOPES, PART 2

> Your Majesty says, "Kill a gentleman, and a gentleman is told off to be killed. Consequently, that gentleman is as good as dead - practically, he is dead - and if he is dead, why not say so? *The Mikado, Act II*, by Gilbert and Sullivan

We found equations for the slopes of the tangents to some simple curves in the last chapter and are now ready to tackle more complicated curves. In the rest of this text, by the way, we shall simplify the language by speaking of the *slope of a curve* when we mean the *slope of the tangent to the curve.*

In this Chapter we shall introduce a quantity, that we call a "dibbl", so small that its square "is as good as" zero - practically, it is zero - and if it is zero, why not say so? The dibbl will make it easier for us to calculate the slope of the tangent to any curve at a point on the curve.

3.1 NEW LANGUAGE FOR CHAPTER 2

We found, in Chapter 2, the slope of a quadratic function $y(x) = x^2$ at a point $x = x_1$. We did this in Eq. **??** by finding the ratio of the change in y to the change in x between two point x_2 and x_1. and then letting $x_2 = x_1$. Writing the last statement in symbols, we have

$$\text{slope} = \frac{y(x_2) - y(x_1)}{x_2 - x_1} \tag{3.1}$$

and then let $x_2 = x_1$. The same procedure should work for any smooth function $y(x)$, but the procedure would become pretty tedious if the function is very complicated, as the next exercises demonstrate.

EXERCISE

3.1 Find the slope of the function $y(x) = x^4$ at the point $x = x_1$ using the method of Eq. 3.1. Show all your work. Hints: (1. See Exercise 1.14. 2. Check your result by noting that when $x_1 = 3$ the value of the slope is 108.)

3.2 You are running around a circular quarter-mile track. The coach tells you that your distance from the starting line is increasing proportionally to the cube of the time. At the end of 1 second you have traveled 3 m.

(a) Write an equation for $s(t)$, the distance traveled, as a function of t. Check your equation by noting that you must have traveled 24 m at the end of 2 seconds.

(b) Your speed at any time t is the rate of change of your distance at time t. Find your speed at any time t. (Here we are writing just t, instead of t_1).

(c). A horse can gallop with a top speed of about 7 m/sec. After how many seconds are you running at a horse's top speed?

(d) Did you have to consider the shape of the track when you answered (a) through (c)?

It turns out that we can simplify the calculation of slope by taking the points x_2 and x_1 to be very close together. This makes sense, because at the end of the calculation we are going to let $x_2 = x_1$. So let's start by simplifying the notation, dropping the subscripts and restating the problem as follows: **Find the slope of the function $y(x)$ at some value of x.**

We will do this by finding the slope of a straight line that intersects $y(x)$ at two points that are so close together that they are **practically, but not quite, the same point**. We write the two points as $y(x)$ and $y(x + dx)$. Since x and $x + dx$ are nearly the same, we say that $x + dx$ is just a *dibbl* away from x.

Also, since we know that $y(x)$ is described by a smooth curve, we also know that the two points on the curve with y-values $y(x)$ and $y(x + dx))$ are just a dibbl away from each other.

We have already said that a dibbl is a quantity so small that we can take its square to be zero "for all practical purposes." So for the rest of this book we shall use "the dibbl equation"

$$dx \times dx = dx^2 = 0 \tag{3.2}$$

We demonstrate the use of the dibbl by recalculating Eq. **??**, the slope of a quadratic at the point $y(x)$. we write:

$$\begin{aligned} \text{slope} &= \frac{y(x + dx) - y(x)}{(x + dx) - x} \qquad (3.3) \\ &= \frac{c(x^2 + 2cxdx + \overbrace{dx^2}^{=0}) - cx^2}{dx} \\ &= \frac{2cx \times \cancel{dx}}{\cancel{dx}} = 2cx \end{aligned}$$

Eq. 3.3 no longer requires the step of setting the point x_2 equal to the point x_1. The dibbl equation makes that step unnecessary.

Note that the slope in Eq. 3.3 is a function of x. It is, in fact, a function of x that was *derived* from the original function $y = cx^2$. Mathematicians, for this reason, call the slope of a function y(x) the *derivative* and write it as a ratio of two dibbls, dy and dx. To summarize, then, for any smooth function $y(x)$ there is a slope function, called the derivative,

$$\frac{dy}{dx} = \frac{y(x+dx) - y(x)}{(x+dx) - x} = \frac{y(x+dx) - y(x)}{dx} \tag{3.4}$$

IMPORTANT NOTE Many textbooks, for historical reason, write the derivative of a function $y(x)$ with a "prime" symbol. So the symbols $y', y(x)', y'(x)$, and $\frac{dy}{dx}$ all mean exactly the same thing. We shall sometimes use the prime symbol in this text, so that you can accustom themselves to both usages.

EXERCISE

3.3 Redo Exercise 3.1, this time using the dibbl method.

3.2 POWERS OF SMALL NUMBERS; THE DIBBL dx

We have introduced the dibbl as a quantity "that is so small that its square is as good as zero". "How small is that?", you might say. So let's get some experience with the squares of small numbers.

EXERCISE

3.4 You may use your calculator for this exercise:
(a) How much is the square of $\frac{1}{5}$? (Hint: If you didn't get $\frac{1}{25}$, ask your teacher to send you back to third grade).
(b) Now take a piece of paper and a ruler marked in centimeters and draw a straight line that is 10 cm long. Divide the line into 5 equal parts by marking little dots ("tick marks") on the line.

Next, divide each division into 5 equal parts, again using tick marks. Into how many divisions have you divided the 10 cm line? Your line should resemble the line in Fig. 3-1.
(c) Does the distance between the little tick marks in Fig 3-1 represent the square of the distance between the large tick marks? (Hint: If you answered "No", go back and redo part (a) of this Exercise). Now complete the following table, using your calculator if you so desire:

FIG 3-1

$x =$	$\frac{1}{5} =$	$.2$	$x^2 =$	$\frac{1}{25}$	$= .04$
$=$	$\frac{1}{50} =$				$=$
$=$	$\frac{1}{500} =$				$=$
$=$	$\frac{1}{5000} =$				$=$
$=$	$\frac{1}{50000} =$				$=$

(d) Are you tired of writing out the zeros in part (c)? Let's do the same calculations in power-of-ten notation, which we will use in all future exercises that involve numbers:

$x =$	$\frac{1}{5} =$	2×10^{-1}	$x^2 =$	$\frac{1}{25}$	$= 4 \times 10^{-2}$
	$\frac{1}{5\times 10} =$				$=$
	$\frac{1}{5\times 10^2} =$				$=$
	$\frac{1}{5\times 10^3} =$				$=$
	$\frac{1}{5\times 10^5} =$				$=$

Exercise 3.4 shows that the square of a fraction means "taking a fraction of a fraction". If the fraction is a very small number then its square may be very small indeed. How small must the number be so that the square "is as good as" zero? In other words, how small is a dibbl? We shall discuss this question further in Chapter 28 when we discuss *limits*.

Our conclusion is that a dibbl is not an ordinary number, but is something that is smaller than any fraction!

We close this section by reminding ourselves why it is difficult to assign a value to $\frac{1}{0}$.

EXERCISE

3.5

(a) Ask your calculator to find a value for $\frac{1}{0}$, and write down the result. If you got a value, you are finished.

(b) If your calculator couldn't do the problem try dividing 1 by a lot of values in the denominator that are close to zero. Then you can try to guess the result. One way to do this is to take a sheet of graph paper, and plot values of $\frac{1}{x}$ for a lot of different fractional values of x. (Don't forget to include negative values). The answer should be roughly half way between the value for the smallest positive value of x and the smallest negative value. What is your result?

(c) If you are in a study group, compare your result with others in your group. Can you all agree on the same value?

(d) What would you say to someone who claims that division by zero is "undefined"? Explain

your answer, referring to your results in part (b).

3.3 dx^2 IS ZERO SIMPLIFIES THE BINOMIAL THEOREM

EXERCISE

3.6 (a). Write down the derivative functions (the slope function, remember?) $\frac{dy}{dx}$ for $y(x) = x, x^2, x^3, x^4$. (You worked these out in Section 3.1).
(b) Guess what the derivative function is for $y(x) = x^n$ where n stands for any positive integer (do not use a specific numerical value for n), and write down your answer.

We now help you work out the answer to Exercise 3.6(b), using the definition of the derivative in Eq. 3.4. The first step is to expand the product $(x + dx)^n$, remembering that all powers of dx higher than the first power are zero. We write:

$$y(x) = (x + dx)^n = \overbrace{(x + dx)(x + dx) \cdots (x + dx)}^{\text{n times}} \tag{3.5}$$

The highest power of x in Eq. 3.5 comes from multiplying all of the $x's$ together. There is only one way to do this, so there is a leading term x^n.

The next term will come from multiplying together $(n - 1)$ $x's$ and one dx. How many such terms are there? That's an easy question, because there are exactly n different $dx's$ in the product in Eq. 3.5. The second term is therefore $nx^{n-1}dx$.

What about the next term? It will have the product of $(n - 2)$ $x's$, two $dx's$ and a numerical coefficient, let's call it a. There will be more terms, involving products of three and even more $dx's$ (if n is bigger than 2). The numerical coefficient that goes with each term is known from the *binomial theorem* that people learn about in pre-calculus courses.

But we don't care about the value of a and all those numerical coefficients because they all involve products of two or more $dx's$ and since dx is a dibbl the product of two or more $dx's$ is zero. The result, then, is $(x + dx)^n = x^n + nx^{n-1}dx$, which is the statement of the binomial theorem for x plus a dibbl taken to any power n.

Eq 3.4 gives, for $y(x) = x^n$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(x^n)}{dx} \\ &= \frac{(x^n + nx^{n-1}dx) - x^n}{dx} \\ &= \frac{nx^{n-1}dx}{dx} = nx^{n-1} \end{aligned} \tag{3.6}$$

Eq 3.6 is often called "the power law".

EXERCISE

3.7 Calculate the derivatives $y' \equiv \frac{dy}{dx}$ ($\equiv$ means that the two expressions mean the same thing), using the method of 3.6 when
(a) $y(x) = x^9$;
(b) $y(x) = 5x^{17}$;
(c) $y(x) = 6x^5 + 4x^4$
Show all of your work.

3.4 "DERIVATIVE" IS A FANCY NAME FOR VELOCITY OR RATE OF CHANGE

EXAMPLE 3.1

Find an expression for the straight line that is tangent to the curve described by $y(x) = \frac{1}{7}x^5$ at the the point $x = 2$. Show your work and graph the result.
Solution:
(1) Let $y_t(x) = a + bx$ be the equation for the straight line. Then our job is to determine the values of the constants a and b.
(2) Since $y_t(x)$ is a tangent, it must have the same slope as $y(x)$ at $x = 2$. The slope of $y_t(x)$ is b (see Subsection 2.1.2). The slope of $y(x)$ at any point x is $\frac{dy(x)}{dx}$
(3) At $x = 2$ $\frac{dy(x)}{dx} = \frac{5}{7}x^4 = \frac{80}{7}$. So $b = \frac{80}{7}$.
(4) Also, $y_t(x)$ and $y(x)$ must be equal at the point of tangency, since they share the same point. So $a + \frac{80}{7}x = \frac{1}{7}x^5$ at $x = 2$, or $a + \frac{160}{7} = \frac{32}{7}$. Thus, $a = -\frac{128}{7}$.
(5) The solution is $y_t(x) = \frac{-128+80x}{7}$. Fig. 3-2 is a graph of the solution showing the tangent line and the curve.

Recall why we are interested in the slopes of tangents to curves. As we pointed out in Chapter 1 (after Fig. 1-3) "the slope of a line on a distance-time graph ... corresponds to velocity". So if we are given a function that relates distance to time, the derivative of that function, being the the slope of the function, corresponds to the velocity as a function of time. The next exercise makes use of that relationship.

EXERCISE

3.8 (a) A rock is hurled upward from a platform that is H meters (Hm) above the ground with an initial velocity v_0 m/sec. We learn in physics that the height of the rock above the ground at time t after it left the platform is

$$h(t) = H + v_0 t - ct^2 \tag{3.7}$$

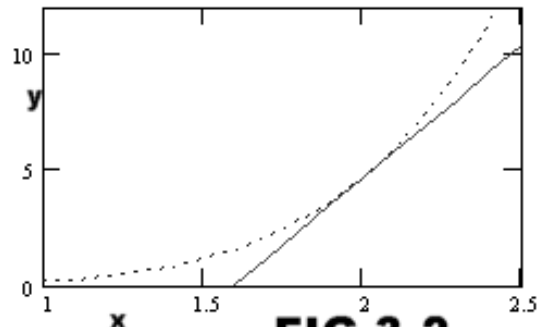

FIG 3-2

where c is a constant. Find the velocity function $v(t)$ of the rock. (Hint: use the method of the previous section. Make sure that you treat "h" and "H" as two different symbols.)
(b) Find an expression for t_m, the time when the velocity function equals zero.
(c) Find an expression for the slope of $h(t)$ at time t_m.
(d) Find an expression for the straight line that is tangent at time t_m to the curve described by $h(t)$ on a distance-time graph.
(e) Find an expression for the maximum value of $h(t)$ as a function of t. (Hint: to check your work, the answer to this part, when $H = 8\ m$; $v_0 = 12\ m/sec$; and $c = 5\ m/sec^2$, is $15.2\ m$.
(f) Make a graph of $h(t)$ on the vertical axis against t on the horizontal axis, using the values in part (e). The horizontal axis should be at least 15 cm long, and the horizontal scale should range from 0 to 4 secs. Calculate $h(t)$ for at least 8 values of t.

3.9 Now let's work out the equation for the tangent line to any power function. So let $y(x) = cx^n$ where c can be any constant and n is any positive integer. Write the tangent line, as before, $y_t(x) = a + bx$.
(a) Find the value of b at an arbitrary point x_0. (Hint: b will be a function of $c, n, and x_0$).
(b) Now find the value of a and write your expression for $y_t(x)$.
(c) Check your result by letting c and n take the values used in the example.

3.10 And, for an algebra brush-up, solve for t by completing the square $at^2 + bt + c = 0$. Show all your work.

3.5 ...AND, IN CLOSING

The "dibbl" quantities $dx, dy, etc.$ "whose squares are as good as zero" actually have a different name in the mathematical literature, where they are known as *infinitesimals* (meaning that they are infinitely small) or *differentials* (as distinguished from differences). They were

first introduced by Gottfried Wilhelm Leibniz (1646 -1716) a German contemporary of England's Sir Isaac Newton (1642 - 1727).. Textbooks often credit Leibniz and Newton with inventing of the calculus. This credit is surely an exaggeration that we shall touch upon again at the end of this book.

Leibniz was aware that his infinitesimals could not represent ordinary numerical quantities. He knew, for example, that any finite number, no matter how large, times an infinitesimal was still an infinitesimal. He probably thought of dx as the distance between two adjacent points on the x-axis of a graph. In this sense, a line passing through two points on a curve that are separated by a distance dx is in fact tangent to the curve at a point.

Problem 3.8 shows that we now know how to find the derivative of any polynomial function. Our next task is to explore some properties of derivatives in order to learn how to find derivatives of more complicated functions.

3.5.1 PUZZLE CORNER NO CREDIT BUT TRY IT ANYHOW

Two cars are approaching each other at a constant (relative) velocity of 60 mph (exact). When the cars are exactly 3 miles apart a very fast fly leaves the bumper of one car and travels toward the other at a speed (over the ground) of 120 mph. Upon reaching the second car the fly immediately reverses direction. This continues until the cars collide. How far did the fly travel?

ANSWER TO THE CHAPTER 2 PUZZLE

Divide the bricks into two piles of 3 and 1 pile of 2. Balance the two piles of 3. If one pile is heavier than the other, the heavy brick is in the heavy pile. Otherwise it is in the pile of 2 bricks and may be found by weighing the two bricks against each other. So assume that one of the piles of 3 is heavy.

Weigh two of the bricks from the heavy pile against each other. Then the either one brick is heavy, or the heavy brick must be the third brick.

Question: Doesn't this answer show how to find 1 heavy brick among 9 bricks in two weighings?