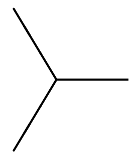
Examination: Quantum Mechanics Around the Clock.



(H Weyl, 1927). Will illustrate for just $N = 3$ "hours". $(\hbar = 1)$

$$\omega \equiv e^{2\pi i/3} = -1/2 + i\sqrt{3}/2 \,, \qquad \Longrightarrow \qquad \omega^3 = 1 \,, \qquad \Longrightarrow \qquad 1 + \omega + \omega^2 = 0 \,.$$

In the 3-dim $|q\rangle$ picture, so that $|q+3\rangle = |q\rangle$, the Hilbert space is spanned by $|0\rangle$, $|1\rangle$, $|2\rangle$, and all operators are 3×3 matrices.

The shift matrix permutes cyclically,

$$U \equiv \omega^{\hat{p}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

$U|q\rangle = |q-1\rangle$; unimodular ($\det U = 1$) ; $U^2 = U^\dagger = U^{-1}$.

The clock matrix reads the hour,

$$V \equiv \omega^{\hat{x}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} .$$

$V|q\rangle = \omega^q |q\rangle$; unimodular; $V^2 = V^\dagger = V^{-1}$.

$U^3 = V^3 = \mathbb{1}$.

Q1: What are the eigenvalues and eigenvectors of V and U?

$\looparrowright$ Weyl's braiding relation: $UV = \omega VU$. It is satisfied by such finite matrices; but *also* for infinite-dimensional ones, with $[\hat{x}, \hat{p}] = i\mathbb{1}$, and now ω modified to $\exp(-i)$.

Q2: Prove this, $e^{\hat{p}} e^{\hat{x}} = e^{-i} e^{\hat{x}} e^{\hat{p}}$.

The Sylvester Finite Fourier Transform matrix,

$$S \equiv \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} ,$$

so $S|q\rangle = \frac{1}{\sqrt{3}} \sum_{r=0}^{2} \omega^{qr} |r\rangle$, unitary, $S^{-1} = S^\dagger$, $S^4 = 1\!\!1$;

$$S^{-1}US = V \qquad \Longrightarrow \qquad SVS^{-1} = U = S^{-1}V^{-1}S.$$

Q3: How/why does S relate to the eigenvectors of U ?

$$\hat{x} = \frac{3}{2\pi i} \log V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

Consequently, (Q4: Why?) ,

$$\hat{p} = \frac{3}{2\pi i} \log U = \frac{3}{2\pi i} S \ \log V \ \ S^{-1} = \begin{pmatrix} 1 & z & z^* \\ z^* & 1 & z \\ z & z^* & 1 \end{pmatrix} = 1 + zU + z^* U^\dagger,$$

where $z \equiv \frac{\omega - 1}{3} = \frac{1}{\omega^2 - 1}$.

Q5: What are $\hat{p}$'s eigenvectors and eigenvalues? How do they relate to those of $\hat{x}$?

Hence,

$$[\hat{x}, \hat{p}] = \begin{pmatrix} 0 & -z & -2z^* \\ z^* & 0 & -z \\ 2z & z^* & 0 \end{pmatrix}.$$

Q6: Check that $-i[\hat{x}, \hat{p}]$ is Hermitean, (as is the infinite-dimensional case covered weeks ago). Note this commutator is *traceless*, as it should be (Why?) for finite-dimensional matrices.

(But this is unlike the $N = \infty$ case, where $-i[\hat{x}, \hat{p}] = 1\!\!1$. Also, this infinite-case commutator is *central*, i.e. it commutes with $\hat{x}$ and $\hat{p}$.)

Q7: For our $N = 3$ case, do $[[\hat{x}, \hat{p}], \hat{x}]$ and/or $[[\hat{x}, \hat{p}], \hat{p}]$ vanish?

↬ This is not the Heisenberg algebra: it is linked to the algebra of Sylvester's "nonions", ultimately $GL(3)$ or $SU(3)$.

► Consider a free particle with $m = 1/2$, so that $\hat{H} = \hat{p}^2$. The time-evolution matrix is $T = e^{-it\hat{H}}$.

Q8: Evaluate $\hat{H}$, its eigenvalues and eigenvectors. What is the ground/vacuum state?

Q9: Compute T. (Hint: $T = Se^{-it\hat{x}^2} S^{-1}$.) Where does $|g\rangle \equiv \frac{|0\rangle + |1\rangle + |2\rangle}{\sqrt{3}}$ go to at all t? Where does any state $|\psi\rangle$ go to at $t = 2\pi$? What is the transition amplitude $(\frac{\langle 0|-\langle 1|+2\langle 2|}{\sqrt{3}})\ T\ |g\rangle$?

Q10: What is $\Delta x \equiv \sqrt{\langle \hat{x}^2\rangle - \langle \hat{x}\rangle^2}$, Δp, and the $\Delta x\ \Delta p$ uncertainty/dispersion of the state $|0\rangle$? of $|g\rangle$? Are these consistent with the Kennard-Robertson inequality:

$$\Delta x \Delta p \geq \frac{1}{2}\ |\langle\psi|[\hat{x},\hat{p}]|\psi\rangle| \quad ?$$

$\curvearrowright$ For generic N, (assuming $\omega \equiv e^{2\pi i/N}$ is a primitive root of unity, so $\omega^{r-s} = 1$ only for $r = s$, as in the case $N = 3$ considered here), the r, s matrix elements of the commutator are (Santhanam & Tekumalla, 1976),

$$[\hat{x},\hat{p}]_{rs} = \frac{r-s}{N} \sum_{k=0}^{N-1} k\omega^{k(r-s)} = \frac{r-s}{\omega^{r-s}-1},$$

or 0 for $r = s$ on the diagonal.

Now, with S&T, take $N \to \infty$, for $\frac{r-s}{N} = x - x'$, to evaluate the infinite-dimensional commutator matrix entries. Rescaling $\hat{x}$ and $\hat{p}$, we have

$$\begin{aligned} -i\langle x|[\hat{x},\hat{p}]|x'\rangle &= -(x-x') \int_{-\infty}^{\infty} dk\ k\ e^{2\pi i k(x-x')} \\ &= -(x-x')\partial_x \delta(x-x') \\ &= \delta(x-x'). \end{aligned}$$

Q11: Prove the last two equalities. What happened to the zeros on the diagonal?

$\rightsquigarrow$ Thus, $[\hat{x},\hat{p}] = i\mathbb{1}$, a traceful, central matrix has emerged.