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Curtright & CZ, JPhysA42 (2009) 485208; JPhysA43 (2010) 445101; PhysRevD83 (2011) 065019; JPhysA44 (2011) 405205.

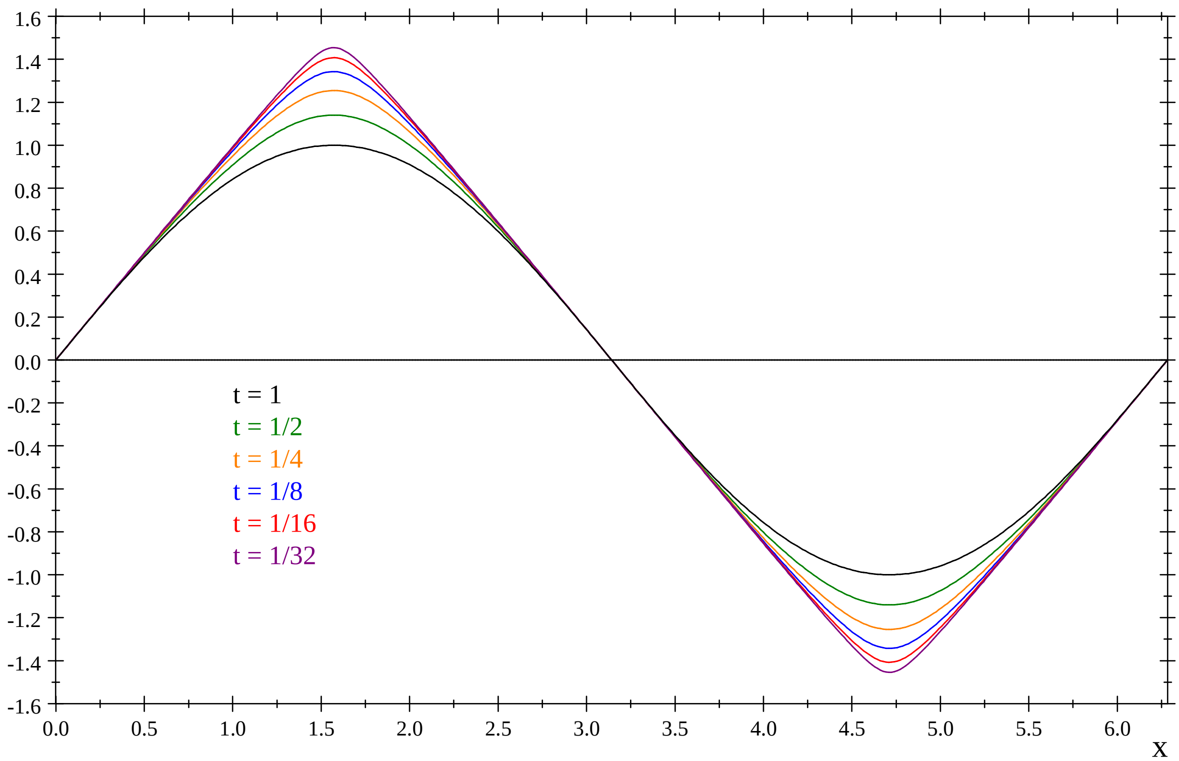
HOLOGRAPHIC INTERPOLATION

SMOOTH DYNAMICS FROM BOUNDARY CONDITIONS

How would you find $\mathrm{rin}(x)$, the functional square root of

$$\boxed{\mathrm{rin}(\mathrm{rin}(x)) = \sin(x)}$$

?


1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
-1.6
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
5.5
6.0
x
t = 1
t = 1/2
t = 1/4
t = 1/8
t = 1/16
t = 1/32

Consider the standard **logistic map** for the special chaotic case $r = 4$,

$$\boxed{x_1 = 4x(1-x) \equiv f_1(x),}$$

where $x \equiv x_0$. Can think of the iteration subscript as a discrete time, and thus the map as a time-translation-invariant one, $x_{t+1} = f_1(x_t) = f_1(f_t(x)) = f_{t+1}(x)$, an associative and commutative composition, $\circlearrowleft$.

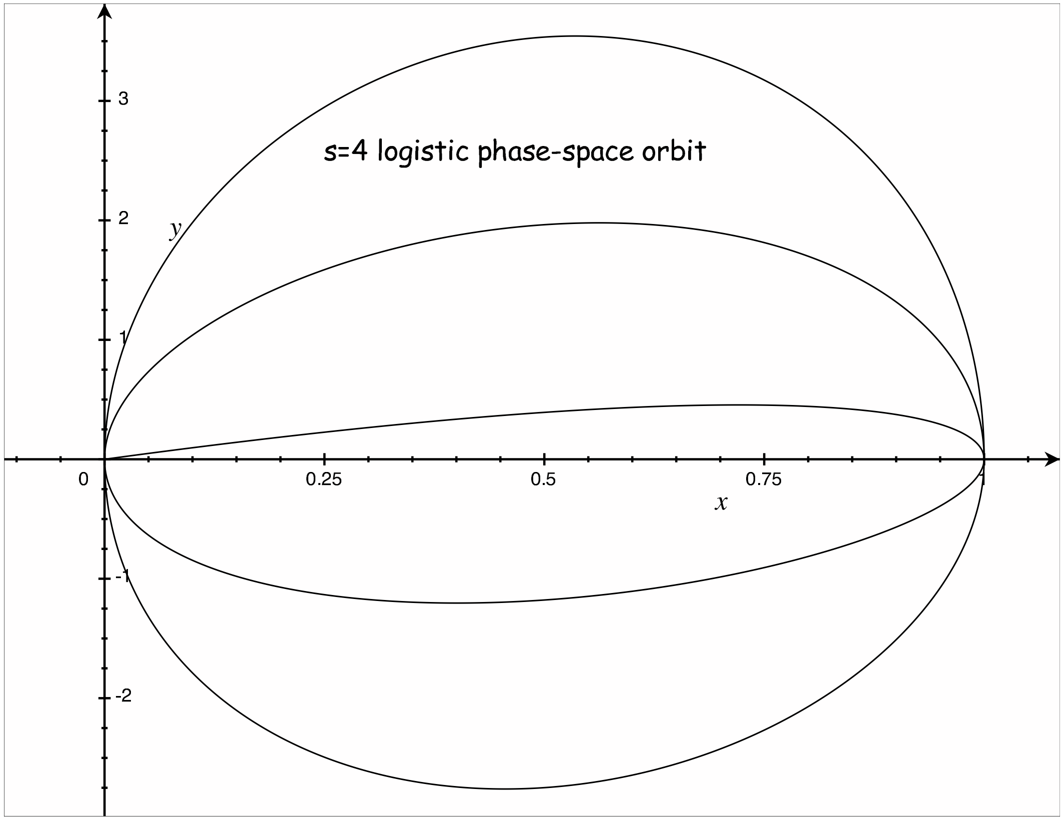
Hemi-heuristically (1870), Ernst Schröder found a closed form solution for **all iterates**, positive or negative,

$$\boxed{f_t(x) = \sin^2(2^t \ \arcsin(\sqrt{x})),}$$

essentially analytic in x.

$\leadsto$ Could thus consider **all** t's, including fractional, negative, continuous and **infinitesimal ones**.

Yields a phase-space orbit of $\dot{f}(t) \equiv v(f(t))$ vs $f(t)$,

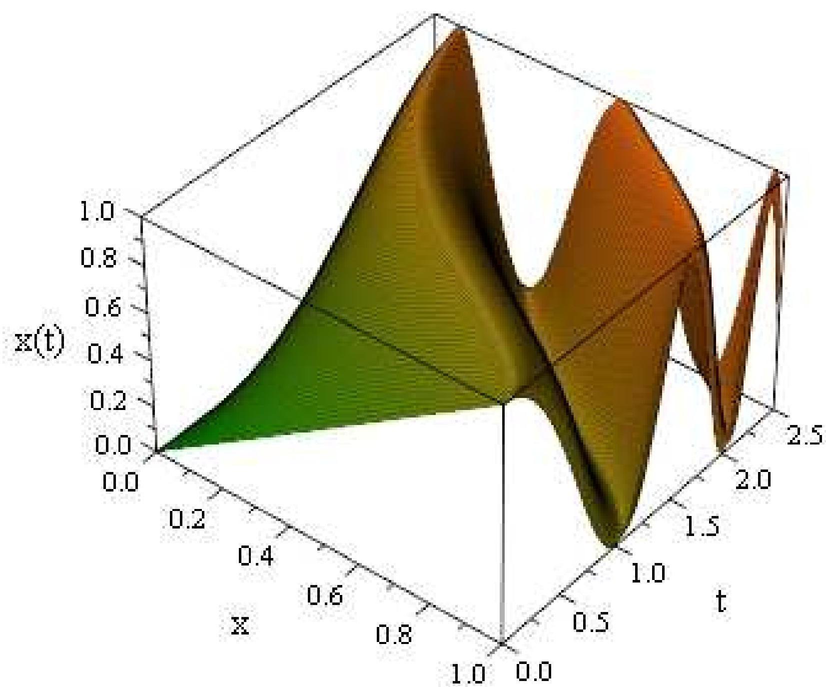


$\leadsto$ Can thus appreciate that this discrete map results out of a continuous hamiltonian evolution driven by a **potential**!

$$\boxed{V(x) = (\ln 4)^2 x(x-1)(n\pi + \arcsin\sqrt{x})^2.}$$

▶ Caveat: actually, a **succession** of deepening potentials at each cycle, $n = (-)^P \lfloor \frac{1+P}{2} \rfloor$. Switchbacks $\looparrowright$ chaos: stretching and folding.

Analogy to inverse scattering: initial and final profiles yield a potential.

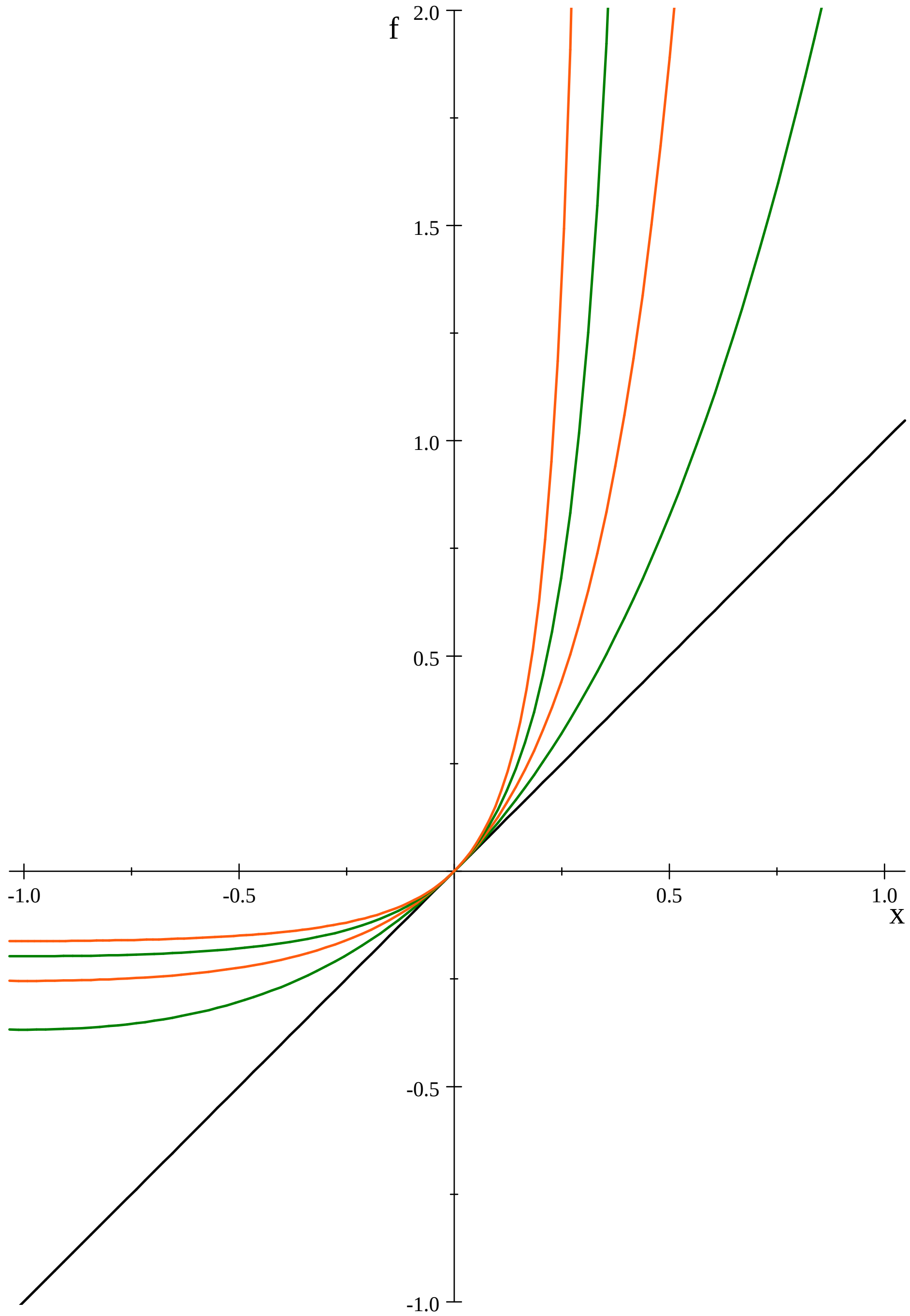


In general, given an evolution profile of a function $x(t)$ for a discrete time interval, from $t = 0$ to $t = 1$, s.t. $x(0) \equiv x$, $x(1) = f_1(x)$, it is straighforward to produce all integral iterates, $\circlearrowleft$, on an **integer lattice of time points**, $t = \cdots, -2, -1, 0, 1, 2, 3, \cdots$: the **splinter**, of the map,

$$
\begin{aligned}
x(2) &= f_1\left(f_1\left(x\right)\right) = f_2\left(x\right) \ , \\
x(n) &= f_1\left(f_1 \cdots \left(f_1\left(x\right)\right)\right) = f_n\left(x\right) \ , \\
x(-1) &= f_1^{-1}\left(x\right) = f_{-1}\left(x\right) \ ,
\end{aligned}
$$

so $x = f_{-1}\left(f_1\left(x\right)\right) = f_1\left(f_{-1}\left(x\right)\right)$, or more generally, $x(k+n) = f_k\left(f_n\left(x\right)\right) = f_n\left(f_k\left(x\right)\right)$, associatively and commutatively.

E.g., for $f_1(x) = x \exp(x)$,

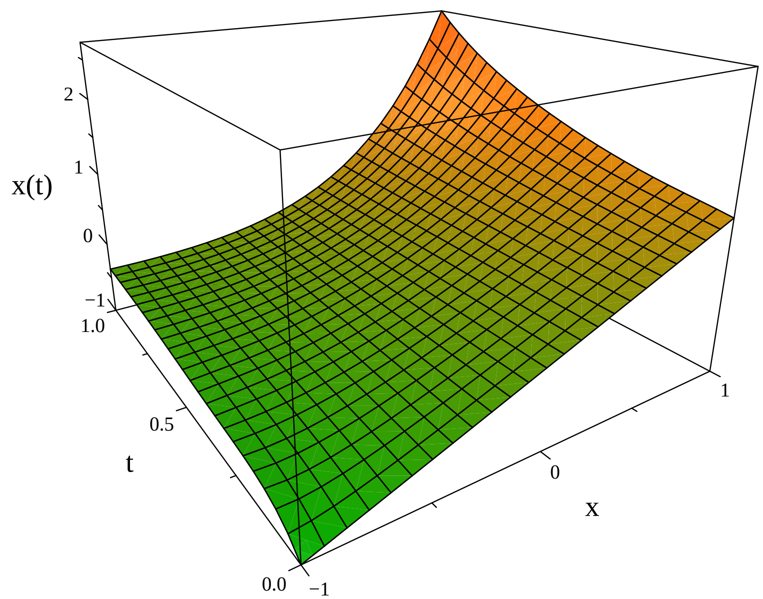


f
2.0
1.5
1.0
0.5
-0.5
-1.0
-1.0
-0.5
0.5
1.0
x

◎ Suppose, however, that, for reasons dictated by a physics context, only an explicit nonlocal **discrete propagation function** $f_1(x)$ such as this is available, but no time-local evolution law is specified.

THE PROBLEM: How does one obtain the complete, **continuous trajectory** $x(t) = f_t(x)$ **without benefit of a local relation**?

E.g., holographic interpolation of $f_1(x) = x\,\exp(x)$:



⫛ Use the elegant construction pioneered by Schröder of an analytic $f_t(x)$ around a **fixed point** of $f_1(x)$.

▶ (Without loss of generality, take the fixed point to be $x = 0$.)

★ THE CONJUGACY FUNCTIONAL EQUATION

Schröder's equation involves the auxiliary function Ψ,

$$\boxed{s\Psi(x) = \Psi(f_1(x)),}$$

for some constant $s \neq 1$. With the origin a fixed point of f_1, i.e., $f_1(0) = 0$, $\leadsto$ $\Psi(0) = 0$, and if $\Psi'(0) \neq 0, \infty$, then $s = f_1'(0)$.

The inverse function satisfies "Poincaré's equation",

$$\Psi^{-1}(sy) = f_1\left(\Psi^{-1}(y)\right).$$

Upon iteration ⥁ of the functional equation, Schröder's Ψ acts on the splinter of x to give

$$s^n\Psi(x) = \Psi(f_n(x)) = \Psi(f_1(f_1 \cdots (f_1(x)))) \ .$$

↬ **This formula naturally yields a continuous interpolation** for all non-integer t,

$$s^t \Psi(x) = \Psi(f_t(x)) \ .$$

⫛ To produce the full, continuous trajectory, solve for Schröder's function $\Psi(x)$, and invert to Ψ^{-1}. This yields $x(t)$ as a functional conjugacy (similarity transform) of the s^t multiplicative map:

$$\boxed{x(t) \equiv f_t(x) = \Psi^{-1}\left(s^t \ \Psi(x)\right).}$$

⇝ In a suitable domain, this trajectory gives the **iteration group** ⥀: the **general iterate for any** t, **analytic around the fixed point** $x = 0$.

✓This solution manifestly satisfies the requisite associative and abelian composition properties for all iterates and inverse iterates. I.e., $f_{t_1+t_2}(x) = f_{t_1}\left(f_{t_2}(x)\right)$, hence $x(t_1 + t_2) = f_{t_1}(x(t_2))$, as required for time-translationally invariant systems.

Some specific cases:

$$
\begin{aligned}
f_2(x) &= \Psi^{-1}\left(s^2\ \Psi(x)\right) = \Psi^{-1}(s\Psi(f_1(x))) = f_1(f_1(x)) \ , \\
f_1(x) = \ & \Psi^{-1}\left(s^1\ \Psi(x)\right) = \Psi^{-1}\left(s^{1/2}\ \Psi\left(\Psi^{-1}\left(s^{1/2}\Psi(x)\right)\right)\right) = f_{1/2}\left(f_{1/2}(x)\right) \\
f_0(x) &\equiv x = \Psi^{-1}\left(s^{-1}\ \Psi\left(\Psi^{-1}\left(s^1\Psi(x)\right)\right)\right) = f_{-1}(f_1(x)) \ ,
\end{aligned}
$$

etc.

⊛ Crucial to note that in the limit $s \to 1$, **all iterates and inverse iterates lose their distinction and degenerate** to the identity map, $f_0(x) = x$, and the method fails as stated. ↬ If $f_1'(0) = 1$, augment $f_1(x)$ in Schröder's equation to $sf_1(x)$, and take the marginal $s \to 1$ limit only **at the very end of the calculation**—if it makes sense to do so.

With the full trajectory $f_t(x)$ now available, the **velocity profile** follows

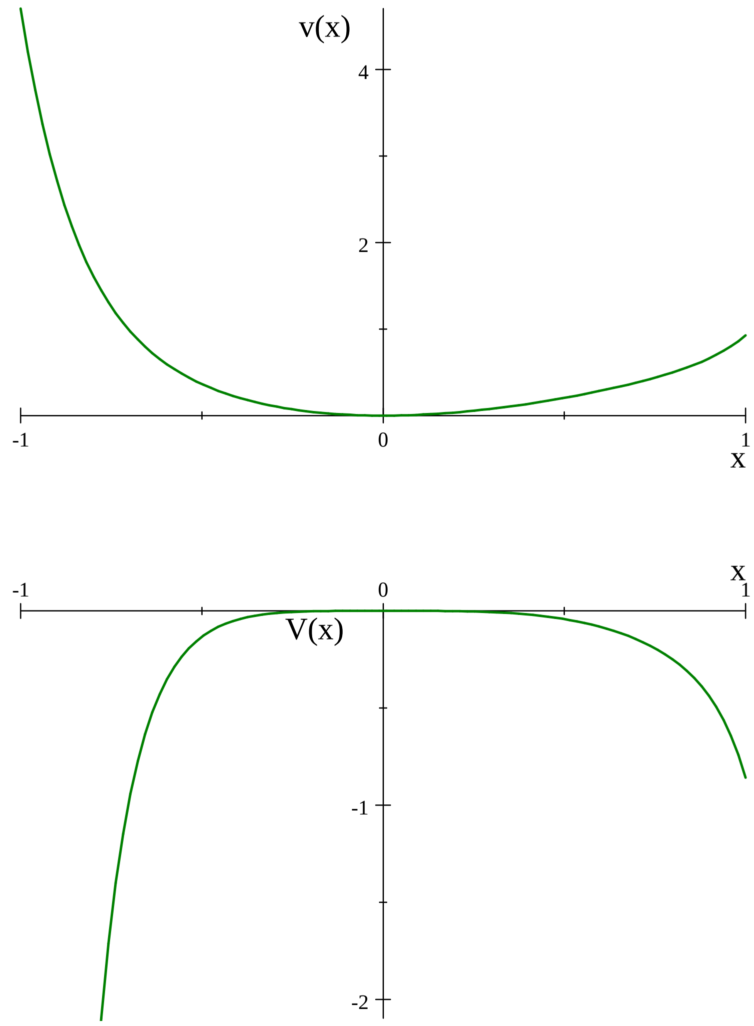
$$v\left(x\left(t\right)\right)=\frac{\partial f_t\left(x\right)}{\partial t}\ ,$$

as an **emergent feature**: $\quad v(x)=\ln s/(\ln\Psi(x))'$.
(Of course, it was not available ab initio to integrate for the trajectory. Had it been, Ψ would have followed by integration! cf. the Gell-Mann – Low renormalization group.)

$\rightsquigarrow$ For motion governed by a Lagrangian, $L=\frac{1}{2}mv^2-V\left(x\right)$, the corresponding **effective potential** $V(x)$, without explicit time dependence, leading to this motion for fixed energy can be determined,

$$V\left(x\right)=-\frac{1}{2}mv^2\left(x\right)+\text{constant.}$$

E.g., for $f_1 = x\exp(x)$, (cf. the Ricker model — salmon!)



$\circledast$ WHAT IS THE MEANING of Ψ? It's but the conjugacy **variable transformation** $w = \Psi(x)$ which **trivializes the action of $f_1(x)$ to a mere scaling** $w \longmapsto sw$, ("rectification") $\rightsquigarrow$ trivial to iterate $\circlearrowleft$ $\forall t$:

$$\begin{array}{ccc} x & \xrightarrow{f_1} & f_1(x) \\ \Psi(x) \downarrow & & \downarrow \Psi(f_1(x)) \\ w & \xrightarrow{s} & sw \end{array}$$

▲ The composite map is then $x \longmapsto \Psi(f_1(x)) = s\Psi(x)$.

$\curvearrowright$ Schröder appreciated his nonlinear conformal conjugacy equation is **hard** to solve analytically in general; but, working backwards and utilizing conformal mappings of monomials and simple trigonometric identities, he found several closed Ψs and their corresponding f_1s, like the logistic map for $r = 4, 2, -2$.

★ **Procedure**: For general functions like $f_1 = x\exp(x)$, ($f_1 = sx\exp(x)$), solve for the Taylor expansion coefficients of Ψ around $x=0$, **recursively** in terms of those of $f_1(x)$, and set $s\to 1$ at the very end. (The nth coefficient of Ψ only depends on those of f_1 of order $\le n$.)

$$\Psi(x) = x - \frac{1}{(s-1)}x^2 + \frac{1}{2}\frac{3s+1}{(s-1)(s^2-1)}x^3 - \frac{1}{6}\frac{16s^3+8s^2+11s+1}{(s-1)(s^2-1)(s^3-1)}x^4 + \frac{1}{24}\frac{125s^6+75s^5+145s^4+146s^3+53s^2+31s+1}{(s-1)(s^2-1)(s^3-1)(s^4-1)}x^5 + O\left(x^6\right).$$

E.g., for $s \equiv e^{\varepsilon}$ near 1, expanding in powers of ε,

$$f_t(x)\Big|_{s=e^{\varepsilon}} = \Psi^{-1}\left(s^t\ \Psi(x)\right)\Big|_{s=e^{\varepsilon}}$$

$$\begin{aligned}
=\ & \left(1+t\varepsilon+O\left(\varepsilon^2\right)\right)x\\
+\ & \left(t+\frac{1}{2}(-1+3t)\,t\varepsilon+O\left(\varepsilon^2\right)\right)x^2\\
+\ & \left(\frac{1}{2}(-1+2t)t+\frac{1}{2}(-1+2t)^2\,t\varepsilon+O\left(\varepsilon^2\right)\right)x^3\\
+\ & \left(\frac{1}{12}\left(5-15t+12t^2\right)t+\frac{1}{12}\left(-7+35t-56t^2+30t^3\right)t\varepsilon+O\left(\varepsilon^2\right)\right)x^4
\end{aligned}$$

$$+\Big(\frac{1}{24}(-2+3t)(5-12t+8t^2)t+\frac{1}{72}(50-315t+673t^2-621t^3+216t^4)\varepsilon$$

$$+O(\varepsilon^2)\Big)x^5+O(x^6).$$

$\rightsquigarrow$

$$v\left(x\right)\Big|_{s=1} = x^2 - 0.5x^3 + 0.416\,67x^4 - 0.416\,67x^5 + 0.445\,83x^6 \\ -0.480\,56x^7 + 0.501\,12x^8 - 0.491\,63x^9 + 0.452\,15x^{10} + O\left(x^{11}\right).$$

• Not singular at $s = 1$!

▶ Positive iterates upward convex with minima at $x = -1$.

⋇ $x = 0$ point of unstable equilibrium in the effective potential.

■ Negative times lead to $f_{-1}(x) = \text{Lambert}W(x)$.

• **Arbitrary functional roots**. Indeed, **full trajectories**, velocities, & potentials: x(noon) & x(now) $\rightsquigarrow$ x(all times)

$\curvearrowright$ **Applications** to Switchback Hamiltonians illuminating chaos.

▲ to field theory holography: AdS/CFT?

✓to finite (conformal and) Renormalization Group $\curvearrowright$:

The finite-RG Gell-Mann–Low (1954) equation is structurally **identical**!

Schröder Functional Conjugacy $\leftrightarrow$ Gell-Mann–Low ($\curvearrowright$Lee)

$$\Psi(x(t)) \qquad \leftrightarrow \qquad G(g(\mu))$$

$$t \qquad \leftrightarrow \qquad \ln \mu$$

$$s \qquad \leftrightarrow \qquad e^d$$

$$\Psi(x(t)) = s^t \Psi(x) \qquad \leftrightarrow \qquad G(g(\mu)) = \mu^d G(g(1))$$

so then $\curvearrowright$

$$g(\mu) = G^{-1}(\mu^d G(g(1)))$$

$$\beta(g) \equiv \frac{dg(\mu)}{d \ln \mu} = \frac{d}{\partial_g \ln G}$$

$$\ln \mu = \int_{g(1)}^{g(\mu)} \frac{dg}{\beta} = \frac{\ln(G(g(\mu))/G(g(1)))}{d},$$

$$\dot{x}(1) = f'(x)\ \dot{x}(0) \quad \text{(Julia eqn)} \qquad \leftrightarrow \qquad \beta(g(e)) = \frac{\partial f}{\partial g(1)}\ \beta(g(1)),$$

$\leadsto$ Extrema of $f(g(1))$ imply zeros of $\beta(g(e))$, before obtaining G.

⋔ Take $g(1) \equiv g$, $G(g) \equiv G$. The RG dictates that scale translations of $g(\mu) \equiv g(\ln\mu, g)$ and functions of it be encoded in motions of the **arbitrary** initial condition $g \equiv g(0, g)$: $\frac{\partial}{\partial \ln\mu} = \beta(g)\frac{\partial}{\partial g}$, $\rightsquigarrow$ the scale translation of $g(\ln\mu, g)$ converts to translation of $\ln G$, since its Taylor expansion around $\ln M = 0$ is

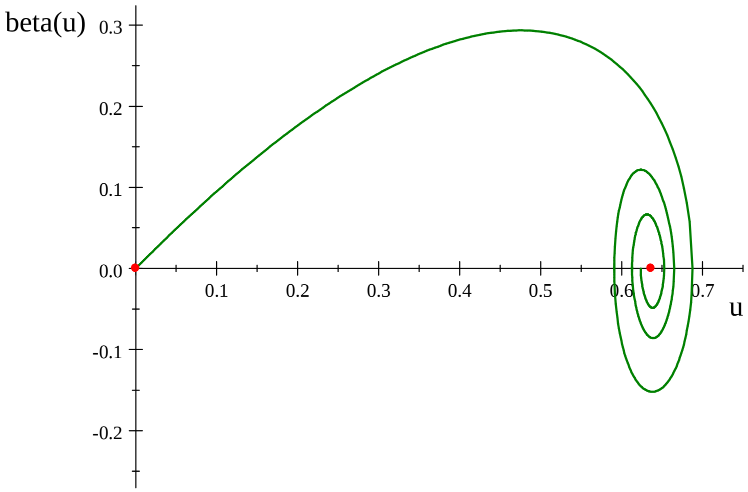
$$g(\mu) = e^{\ln\mu \ \frac{\partial}{\partial \ln M}} \ g(M)\Big|_{M=1} = e^{\ln\mu \ \beta(g)\frac{\partial}{\partial g}} \ g = e^{\ln\mu \ d \ \frac{\partial}{\partial \ln G}} \ g =$$

$$= e^{\ln\mu \ d \ \frac{\partial}{\partial \ln G}} \ G^{-1}(G) = G^{-1}(\mu^d G(g)),$$

the integrated RG, from a mere translation of the variable $\ln G$ by $d \ln\mu$!

⊠ Now, suppose a "step-scaling function" f is obtainable, e.g. from lattice simulations, $g(e) = f(g(1))$. From this function, through our procedure (holographic interpolation of Schröder's eqn), the full $g(\mu)$ and $\beta(\mu)$ can be reconstructed.

(One might further exclude aditional solutions based on **periodic** functions: for any soln $G(g(e))$, there would be the whole (but noninvertible) family of solns $G(g)F(\ln G(g))$ for any periodic fctn F with period d.)



✓ An exotic new feature: When G^{-1} involves a **periodic function**, and a logarithmic function converting multiplication into shifts, $g(\mu) = G^{-1}(\mu G(g))$ yields a **limit cycle**, i.e. periodicity of the physics in the logarithm of the scale μ. Such situations do actually occur in physics!

⋇ E.g., in the "Russian doll" superconductivity model of LeClair, Roman, & Sierra: $G^{-1} = \tan\log$, so that

$$\boxed{g(\mu) = \tan(\ln\mu + \arctan g).}$$

⇝ **The physics repeats itself cyclically** in self-similar modules.
⥁ Spin-glasses exemplify chaotic renormalization.