Cosmas Zachos Varna June 19, 2007

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A CLASSICAL BOUND ON QUANTUM ENTROPY

$$\boxed{0 \leq S_q \leq \ln\left(\frac{e\sigma^2}{2\hbar}\right)}$$

involving the variance σ^2 in phase space of the **classical** limit distribution of a given **quantum** system. No Hamiltonian required.

Intuitively plausible: $\hbar \to 0$ information forfeiture, augmenting ignorance.

$\rightsquigarrow$ A fortiori, this further bounds the corresponding information-theoretical generalizations of the quantum entropy proposed by Rényi.

Black Hole entropic behavior: collective flow of information in need of robust estimates through gross geometrical and semiclassical features of the system—instead of toilsome detailed accounts of subtler quantum states.

$\rightsquigarrow$ **Combines** the upper bound for the entropy of classical continuous distributions (Shannon, 1949) with the classical limit of intricate quantum systems in phase space (Braunss 1994), which tracks the information loss involved in smearing away quantum effects: The quantum entropy of a system is majorized by that of its 'ignorant' classical limit.

• **Illustrated** by the elementary physics paradigm of a thermal bath of oscillator excitations of one degree of freedom: its phase-space representation is a (maximal entropy $\sim$ chaos) Gaussian.

• Extension to arbitrary degrees of freedom and tighter bounds according to the circumstances of physical applications are conceptually straightforward.

SHANNON INFORMATION ENTROPY

For a continuous distribution function $f(x,p)$ in phase space, the **classical** information entropy is

$$S_{cl} = -\int dxdp\ f\ \ln(f).$$

Given a $f(x,p)$, without loss of generality centered at the origin, normalized, $\int dxdpf = 1$, and
with a given variance, $\sigma^2 = \langle x^2 + p^2 \rangle = \int dxdpf(x^2 + p^2)$,
$\rightsquigarrow$ elementary constrained variation of this $S_{cl}[f]$ w.r.t. f, $\rightsquigarrow$
it is maximized by the Gaussian, $f_g = \exp(-(x^2 + p^2)/\sigma^2)/\sigma^2\pi$, to $S_{g\ cl} = 1 + \ln(\pi\sigma^2)$.
• A Gaussian represents maximal disorder and minimal information.
In thermodynamics, least dispersal energy would be available.

$\rightsquigarrow$ Shannon's inequality,

$$\boxed{S_{cl} \leq \ln(\pi e \sigma^2)\ ,}$$

an **upper bound on the lack of information**.

• In general, S_{cl} is unbounded above: it diverges for delocalized distributions ($\sigma \to \infty$), containing no information. In contrast to the Boltzmann-Gibbs entropy, it is also unbounded below, given ultralocalized peaked distributions ($\sigma \to 0$), which reflect complete order and information.

BOLTZMANN GIBBS QUANTUM ENTROPY

In quantum mechanics, the sum over all states is given by the standard von Neumann entropy for a density matrix ρ,

$$0 \leq S_q = -\mathrm{Tr}\ \rho\ \ln \rho = -\langle \ln \rho \rangle\ .$$

$\Rrightarrow$ Transcribes in phase space through the Wigner transition map to

$$\boxed{0 \leq S_q = -\int dx dp\ f\ \ln_\star(hf)\ ,}$$

where Groenewold's (1946) $\star$-product,

$$\star\ \equiv\ e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)}\ ,$$

serves to **define** $\star$-functions, such as this $\star$-logarithm, e.g., through $\star$-power expansions,

$$\ln_\star(hf) \equiv -\sum_{n=1}^{\infty} \frac{(1-hf)_\star^n}{n}\ .$$

Braunss has argued that, for $S_q + \ln h \to S_{cl}$ as Planck's $\hbar \to 0$,

$$\boxed{0 \leq S_q \leq S_{cl} - \ln h \ .}$$

• The logarithmic offset term relying on the Planck constant h accounts for the **scale** of the phase-space cell area $dxdp$.
This scale, h, divides $dxdp$ and multiplies f, to to yield dimensionless entities which preserve 'probability', in the Wigner transition map from the density matrix ρ to the Wigner Function f.

• E.g., for a pure state,

$$f(x,p) = \frac{1}{h} \int dy \ \psi^* \left(x - \frac{y}{2} \right) \ e^{-iyp/\hbar} \ \psi \left(x + \frac{y}{2} \right) \ .$$

The classical limit usually entails activity of phase-space variables much larger than $\hbar$ $\rightsquigarrow$ Scale these variables down to scales matched to such activity. Comparing quantum and classical entropies relies on this offset.

⋔ The upper bound in this Braunss inequality reflects the **loss of quantum information** involved in the smearing implicit in the classical limit. (Sacrifice of the resolution needed to access the uncertainty.)

$\leadsto$ Combined with Shannon's bound, this now amounts to

$$\boxed{0 \leq S_q \leq \ln\left(\frac{e\sigma^2}{2\hbar}\right)},$$

i.e., the entropy is bounded above by an expression involving the variance of the corresponding **classical limit distribution function**.

$\circledast$ Readily generalizes to multidimensional phase space, and contexts where more information (e.g., on asymmetric variances) happens to be available, or refinement desired. (Bekenstein bound.)

• The quantum entropy is recognized as an expansion

$$S_q = \sum_{n=1}^{\infty} \frac{\langle (1-\rho)^n \rangle}{n} = \sum_{n=1}^{\infty} \frac{\langle (1-hf)_\star^n \rangle}{n}.$$

The leading term, $n = 1$, $1 - \mathrm{Tr}\rho^2 = \langle 1 - hf \rangle$, is the **impurity**, often referred to as linear entropy. Like the entropy itself, it **vanishes for a pure state**, for which $\rho^2 = \rho$, or, equivalently, $f \star f = f/h$.

$\leadsto$ Each term in the expansion projects out ρ, or $\star hf$, respectively: **pure states saturate the lower bound on S_q**.

RÉNYI ENTROPY

A likewise additive (extensive) generalization of the quantum entropy is the Rényi entropy,

$$R_\alpha = \frac{1}{1-\alpha} \ \ln\langle \rho^{\alpha-1}\rangle = \frac{1}{1-\alpha} \ \ln\left(\int \frac{dxdp}{h}(hf)^\alpha_\star\right) \ .$$

• The limit $\alpha \to 1$ yields $R_1 = S_q$; and the impurity is $1 - \exp(-R_2)$.

For continuous distributions (infinity of components) discussed here, R_0 is divergent.

⋇ For $\alpha \geq 1$, $R_\alpha \geq R_{\alpha+1}$, so $S_q \geq R_\alpha$, and it is also bounded below by 0,

$$S_q \geq R_\alpha \geq R_{\alpha+1} \geq 0 \ .$$

↬ A fortiori, the Rényi entropy is also **bounded by the same bound**.

GAUSSIAN ILLUSTRATION

Consider the Gaussian Wigner Function of **arbitrary** half-variance E,

$$f(x,p,E) = \frac{e^{-\frac{x^2+p^2}{2E}}}{2\pi E} = e^{-\frac{x^2+p^2}{2E} - \ln(2\pi E)}.$$

This happens to be the phase-space Wigner transform of a Maxwell-Boltzmann thermal distribution for a harmonic oscillator, in suitably rescaled units, normalized properly to unity, and with mean energy $E = \langle \frac{x^2+p^2}{2} \rangle$.

Calculation of the entropy of this distribution, is, of course, a freshman physics problem; review its independent phase-space derivation, evaluate S_q directly.

For $E = \hbar/2$, the distribution reduces to just f_0, the Wigner Function for a **pure state** (the ground state of the harmonic oscillator). $\looparrowright$

$$f_0 \star f_0 = \frac{f_0}{h} ,$$

$\rightsquigarrow$ f_0 is $\star$-orthogonal to each of the terms in the sum, and hence $S_q = 0$, indicating saturation of the maximum possible information content. Trivially, $0 < \ln(e/2) = 1 - \ln 2 \sim 0.307$.

For generic width E, the Wigner Function f is not that of a pure state, but **it still happens to always amount to** a $\star$-exponential ($e_\star^a \equiv 1 + a + a\star a/2! + a\star a\star a/3! + \ldots$) as well,

$$hf = e^{-\frac{x^2+p^2}{2E} + \ln(\hbar/E)} = e_\star^{-\frac{\beta}{2\hbar}(x^2+p^2) + \ln(\frac{\hbar}{E}\cosh(\beta/\hbar))},$$

where an "inverse temperature" variable $\beta(E,\hbar)$ is useful to define,

$$\tanh(\beta/2) \equiv \frac{\hbar}{2E} \leq 1 \qquad \Longrightarrow \qquad \beta = \ln\frac{E+\hbar/2}{E-\hbar/2} \ .$$

(The above pure state f_0 corresponds to zero temperature, $\beta = \infty$.)

Since $\star$-functions, by virtue of their $\star$-expansions, obey the **same** functional relations as their non-$\star$ analogs, inverting the $\star$-exponential through the $\star$-logarithm and integrating yields directly the standard thermal physics result:

$$\boxed{S_q(E,\hbar) = \frac{E}{\hbar}\ln\left(\frac{2E+\hbar}{2E-\hbar}\right) + \frac{1}{2}\ln\left((\frac{E}{\hbar})^2 - \frac{1}{4}\right) = \frac{\beta}{2}\coth(\beta/2) - \ln(2\sinh(\beta/2)).}$$

↬ **monotonically nondecreasing** function of E, attaining the lower bound 0 for the pure state $E \to \hbar/2$ ($\beta \to \infty$, zero temperature).

The classical limit, $\hbar \to 0$ ($\beta \to 0$, infinite temperature) thus follows,

$$S_q \to 1 + \ln(E/\hbar) = \ln(\pi e 2E) - \ln h = S_{cl}(E) - \ln h \ .$$

Explicitly seen to bound the expression for all E; saturating it for large $E >> \hbar$, in accordance with Braunss' bound. I.e., the upper bound is **saturated** for Gaussian quantum Wigner functions with $\sigma^2 >> \hbar$.

⋇ The region $E < \hbar/2$, corresponding to ultralocalized spikes excluded by the uncertainty principle, was **not allowed** by the above derivation method, since, in this region, no $\star$-**Gaussian can be found** to represent the Gaussian. (It would amount to complex β and S_q, linked to thermal expectations of the oscillator parity operator.)

$\star$-powers of the Gaussian are also straightforward to take, and thus the Rényi entropies can also be readily computed:

$$R_\alpha = \frac{1}{1-\alpha} \ln \left(\frac{(2\sinh(\beta/2))^\alpha}{2\sinh(\alpha\beta/2))} \right) = \frac{1}{\alpha-1} \ln \left(\left(\frac{E}{\hbar} + \frac{1}{2} \right)^\alpha - \left(\frac{E}{\hbar} - \frac{1}{2} \right)^\alpha \right).$$

• $\alpha \to 1$ checks with the above $R_1 \to S_q$. Also, in the pure state limit, $E = \hbar/2$, it is evident that $R_\alpha = 0$ checks for all $\alpha \geq 1$.
(For $\alpha > 1$ and the small disallowed values $E < \hbar/2$, $\quad R_\alpha < 0$.)

R_α is also a nondecreasing function of E; and a nonincreasing function of α. Up to an additive, α-dependent constant, the classical limit is identical to that for the entropy itself,

$$\boxed{R_\alpha \to \tfrac{\ln\alpha}{\alpha-1} + \ln(E/\hbar)} \ .$$

in agreement with the classical result.

It may well be that specific αs could provide more detailed or practical measures of complexity in holographic BH physics with sparse information available: gravitational physics confronting quantum randomness. Or when the Compton wavelength is invisible behind its own Schwarzschild horizon!

▲ **If a specific quantum Hamiltonian were actually available** for the system in question (rare), then the classical limit of the entropy of the system would be straightforward $\hookrightarrow$ our inequality would not be that powerful, since the classical entropy itself would be at hand, in general lower than the Shannon bound.
For such simple systems, the upper-bounding classical entropy would result out of the phase-space partition function specified by the corresponding classical hamiltonian (the Weyl symbol of the quantum hamiltonian).

$\rightsquigarrow$ Illustrated explicitly by hamiltonians which are positive N-th powers of the oscillator hamiltonian,

$$\boxed{f_{cl} \propto \exp(-\left(\frac{x^2+p^2}{2E}\right)^N).}$$

By standard thermodynamic evaluation, the bounding classical entropy reduces to just the Shannon entropy,

$$\boxed{S_{cl} = \frac{1}{N} + \ln\left(2\pi E\Gamma(1+\frac{1}{N})\right),}$$

lower than the Shannon bound, $1 + \ln\left(\pi E \frac{\Gamma(1+2/N)}{\Gamma(1+1/N)}\right)$.