The SU(5) Grand Unified Theory

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Abstract

The SU(5) grand unified model of Howard Georgi and S. L. Glashow was among the first attempts to embed the standard model in a larger gauge group and explain some of the standard model's arbitrary features. This model based on SU(5) henceforth called the Georgi-Glashow model has met with partial success and has given us some interesting insights and predictions. I briefly review the main features of the model based on my reading of the topic.

1 Introduction

The current theory of the strong, weak and electromagnetic interactions is based on the group $SU_c(3) \times SU_L(2) \times U_Y(1) \equiv G_s$. We believe that there is a spontaneous symmetry breaking (SSB) at around $100\,GeV$ breaking $SU_L(2) \times U_Y(1) \rightarrow U_{em}(1)$ via the Higgs-Anderson mechanism. This theory prosaically referred to as the 'Standard Model' (SM) has been tremendously successful in describing and interpreting almost all phenomena encountered to date in high energy physics (ignoring neutrinos). But despite its phenomenal successes (See for example [1]) the SM is considered by its practitioners as an incomplete theory with many free parameters that will be explained by a more fundamental theory of the interactions. Some of the unsatisfactory features in the SM are [2],

(a) The pattern of groups and representations is complicated and arbitrary.

(b) The strong, weak, and electromagnetic fine structure constants are not related in any fundamental way and the Weinberg angle cannot be calculated in SM.

(c) The question of charge quantization is left unanswered.

(d) The particle masses are arbitrary since we do not know how to fix the Yukawa couplings.

(e) Gravity is not included in the SM.

One attempt to explain and relate some of the arbitrary parameters in the SM is through *grand unified theories* (GUTs) [2]. The idea of GUTs is to embed the SM group G_s into a larger group G and try to relate the previously arbitrary features of SM through the additional symmetries present. For instance if G is *simple* then it has only one coupling constant before SSB and we may try to relate the SM couplings and the electro weak angle through renormalization group flow. Then the idea is that G undergoes SSB (sometimes multiple) to G_s,

$$G \quad \longrightarrow \quad SU_c(3) \times SU_L(2) \times U_Y(1) \;\equiv\; G_s \quad \longrightarrow \quad SU_c(3) \times U_{em}(1) \tag{1}$$

Currently the most interesting candidates for G are $SU(5)$, $SO(10)$, E_6 and the semi-simple group $SU_c(3) \times SU_L(3) \times SU_R(3)$. In this report we look at a GUT model based on the simple group $SU(5)$ [3]. In section 2 we briefly review the formalism of $SU(n)$ Lie algebra and in section 3 we introduce the $SU(5)$ Georgi-Glashow model. Section 4 examines some of the model's features and section 5 is the conclusion.

2 SU(n) Lie algebras

The $SU(n)$ Lie group is the the set of $n \times n$ unitary matrices with determinant one . A general $SU(n)$ transformation may be written as

$$\mathcal{U} \;=\; Exp\left[-i\Sigma_{k=1}^{n^2-1}\,\beta^p L^p\right] \tag{2}$$

where L^p are the n^2-1 generators of the $SU(n)$ Lie algebra. The generators are chosen to be Hermitian and traceless. The L^p may be normalized so that $trace(L^a L^b) \;=\; \delta_{ab}/2$ (See for example [4]). $SU(n)$ has $n-1$ Cartan generators and is hence of rank $n-1$. This means that we can simultaneously diagonalize $n-1$ generators. To take an example, $SU(3)$ is of rank 2 and we take the Cartan generators as the hypercharge (Y) and the third component of the isospin (I_3).

Any vector $\psi_j = (\psi_1, \psi_2, \psi_3, ...)$ in $\mathcal{C}_n$ is mapped as

$$\psi_j \;\rightarrow\; \psi_j^{'} = \mathcal{U}_{ij}\,\psi_j \tag{3}$$

The ψ_js form the basis for what is called the $fundamental$ representation of $SU(n)$ and is denoted by $\mathbf{n}$. The conjugate representation is defined by the ψ^js and is denoted $\mathbf{n}^*$. It transforms as

$$\psi^{'\;j} = \mathcal{U}_i^j\,\psi^i \tag{4}$$

We may also construct higher rank tensor representations for $SU(n)$ that transform as

$$\psi^{'\;i_1\,i_2\,i_3\,...}_{j_1\,j_2\,j_3\,.....} = (\mathcal{U}^{i_1}_{l_1}\mathcal{U}^{i_2}_{l_2}....)(\mathcal{U}^{k_1}_{j_1}\mathcal{U}^{k_2}_{j_2}....)\,\psi^{l_1\,l_2\,l_3\,.....}_{k_1\,k_2\,k_3\,.....} \tag{5}$$

For the $SU(n)$ gauge theory there are n^2-1 Hermitian gauge fields A^i.

The covariant derivatives for $\mathbf{n}^*$ and $\mathbf{n}$ are defined in the usual way,

$$(D_\mu\psi)^a \;=\; [\partial_\mu\delta^a_b - ig(\mathbf{A}_\mu.\mathbf{L}(\mathbf{n}^*)\,)^a_b]\,\psi^b \tag{6}$$

$$(D_\mu\chi)_a \;=\; \left[\partial_\mu\delta^b_a - ig(\mathbf{A}_\mu.\mathbf{L}(\mathbf{n})\,)^b_a\right]\chi_b \tag{7}$$

In the case of $SU(5)$ we use the labeling $i = (\alpha, r)$ for a representation where the index $\alpha\,\epsilon\,1,2,3$ denotes the $SU(3)$ content and $r\,\epsilon\,4,5$ denotes the $SU(2)$ index. One other point to note is that the irreducible representations in the decomposition of the product of representations may be readily found using the method of Young tableaux [5]. For example in the case of $SU(5)$,

$$\mathbf{5}^* \times \mathbf{10} \;\;=\;\; \mathbf{5} \;+\; \mathbf{45} \tag{8}$$

We are now ready to introduce the Georgi-Glashow model.

3 Georgi-Glashow model

The SM group G_s is a rank 4 group and hence the GUT group G must be of atleast rank 4. There are exactly nine rank 4 local Lie groups that involve only one coupling strength. It was argued by Georgi and Glashow in 1974 that among these only the $SU(5)$ has the most desired properties to form G [3]. One strong reason is that among the nine groups only $SU(5)$ and $SU(3)\times SU(3)$ have complex representations. Among them $SU(3)\times SU(3)$ can be eliminated since it cannot accommodate both integrally and fractionally charged particles.

The matter left handed (LH) fields in SM are in the representations $SU_c(3)\times SU_L(2)$,

$$(u_\alpha, d_\alpha)_L : (\mathbf{3},\mathbf{2}) \ ; \ (\nu_e, e^-)_L : (\mathbf{1},\mathbf{2}) \ ; \ u_L^{c\,\alpha}\,,\, d_L^{c\,\alpha} : (\mathbf{3}^*,\mathbf{1}) \ ; \ e_L^+ : (\mathbf{1},\mathbf{1}) \tag{9}$$

where the superscript c denotes charge conjugation. The simplest realization of the model incorporates the 15 LH fields in the representations,

$$\psi_i \quad : \quad \mathbf{5} \ = \ (\mathbf{3},\mathbf{1},-\mathbf{1/3}) + (\mathbf{1},\mathbf{2},\mathbf{1/2}) \qquad ; \quad Fundamental\ \ representation \tag{10}$$

$$\psi^i \quad : \quad \mathbf{5}^* \ = \ (\mathbf{3}^*,\mathbf{1},\mathbf{1/3}) + (\mathbf{1},\mathbf{2},-\mathbf{1/2}) \qquad ; \quad Conjugate\ \ representation \tag{11}$$

$$\psi_{ij} \quad : \quad \mathbf{5}\otimes_{\mathbf{A}}\mathbf{5} \equiv \mathbf{10} \ = \ (\mathbf{3}^*,\mathbf{1},-\mathbf{2/3}) + (\mathbf{3},\mathbf{2},\mathbf{1/6}) + (\mathbf{1},\mathbf{1},\mathbf{1}) \qquad ; \quad Antisymm.\ rep. \tag{12}$$

Comparing (9) with (10), (11), and (12) shows that the SM fields of a generation may be snugly accomodated in $\mathbf{5}^*+\mathbf{10}$. To make the above a little more explicit, a possible representation of $\mathbf{5}^*$ ignoring Cabibo type mixing may be,

$$\mathbf{5}^* \ : \ \psi_L \ = \ \left(d_1^c,\, d_2^c,\, d_3^c,\, e^-, -\nu_e\right)_L \tag{13}$$

The $SU(5)$ has 24 generators represented by generalized Gell-Mann matrices. The 24 gauge bosons A_b^a transform according to the adjoint representation, decomposing as,

$$\mathbf{24} \ = \ (\mathbf{8},\mathbf{1},\mathbf{0}) + (\mathbf{1},\mathbf{3},\mathbf{0}) + (\mathbf{1},\mathbf{1},\mathbf{0}) + (\mathbf{3},\mathbf{2}^*,-\mathbf{5/6}) + (\mathbf{3}^*,\mathbf{2},+\mathbf{5/6}) \tag{14}$$

with the identification

$$G_\beta^\alpha \ : (\mathbf{8},\mathbf{1},\mathbf{0}) \ ; \ W^\pm, W^0 \ : (\mathbf{1},\mathbf{3},\mathbf{0}) \ ; \ B \ : (\mathbf{1},\mathbf{1},\mathbf{0}) \ ; \ A_\alpha^r : (\mathbf{3},\mathbf{2}^*,-\mathbf{5/6}) \ ; \ A_r^\alpha : (\mathbf{3}^*,\mathbf{2},+\mathbf{5/6}) \tag{15}$$

It is seen that apart from the 12 gauge bosons in the SM there are 12 new Baryon-Lepton number violating gauge bosons A_α^r and A_r^α that carry both flavor and color. They are traditionally denoted as,

$$A_\alpha^r \ \equiv \ (X_\alpha, Y_\alpha) \quad ; \quad A_r^\alpha \ \equiv \ (X_\alpha, Y_\alpha)^T \tag{16}$$

Using the relation for the charge operator acting on a representation,

$$\hat{Q}(\psi_q^p) \ = \ Q_q \ - \ Q_p \tag{17}$$

we deduce the charges of the X and Y gauge bosons as,

$$Q_X \ = \ -4/3 \quad ; \quad Q_Y \ = \ -1/3 \tag{18}$$

The X and Y gauge bosons are also sometimes referred to as di-quark and lepto-quark gauge bosons to denote the processes they mediate. One consequence of these extra bosons is that protons may undergo decay and this places strong experimental constraints on the X and Y gauge boson masses. The proton life time may be calculated in the Georgi-Glashow model and it is found that [2]

$$\tau_p \sim \frac{1}{\alpha_{(5)}^2}\frac{M_X^4}{m_p^5} \tag{19}$$

where $\alpha_{(5)}$ is the $SU(5)$ fine structure constant and M_X and m_p are the gauge boson and proton masses. The observational constraint $\tau_p \geq 10^{30}$yrs gives $M_X \gtrsim 10^{14}$GeV. In the next section we further explore the phenomenology of the Georgi-Glashow model.

4 Phenomenology of SU(5) GUTs

The phenomenological implications of the Georgi-Glashow model dictates how viable it is as a theory of the SM interactions. So let us explore some of the features of the $SU(5)$ model. We follow the treatment in [5].

Charge quantization is automatic in the $SU(5)$ scheme since the group is simple and hence the charge generators have discrete eigenvalues. In the $SU(5)$ GUT the charge operator must be some linear combination of the Cartan generators. Since the charge operator commutes with $SU_c(3)$ generators (since gluons carry no charge) it must be of the form,

$$Q \;=\; I_3 + \frac{Y}{2} \;=\; I_3 + \; k\, I_0 \tag{20}$$

where I_3 and I_0 are the Cartan generators belonging to $SU(2)$ and $U(1)$. The normalization of the Cartan generators are fixed by the commutation relations but we also require consistency with (13),

$$Y(\mathbf{5}) \;=\; (-2/3, -2/3, -2/3, 1, 1) \tag{21}$$

fixing the constant in (20) as $k = -(5/3)^{1/2}$. This yields the charge operator matrix,

$$\hat{Q}(\psi_i) \;=\; Diag\,(-1/3, -1/3, -1/3, 1, 0) \tag{22}$$

Since the generators must be traceless we have from above the interpretation that,

$$3\,Q_d \;+\; Q_{e^+} \;=\; 0 \tag{23}$$

Thus the reason that the quarks carry 1/3 the charge of the leptons is because of the fact that they come in 3 colors. This is a remarkable explanation for the charge assignments in the SM that relates color and hypercharge.

Next we turn to the the question of anomalies in the Georgi-Glashow model. The anomaly in any representation is proportional to [6],

$$\mathfrak{D}^{abc} \;=\; tr\left[\,\{T_R^a, T_R^b\}T_R^c\right] \;=\; \frac{1}{2}\,A(R)\,d^{abc} \tag{24}$$

where d^{abc} is defined by the relation

$$\{L^a, L^b\} \;=\; 2\,d^{abc}L^c \tag{25}$$

Since $A(R)$ is independent of the generators we may use any simple generator to calculate it. Choosing the generators as the charge operator (20), we have

$$\frac{A(\mathbf{5}^*)}{A(\mathbf{10})} = \frac{tr\, Q^3(\psi^i)}{tr\, Q^3(\psi_{ij})} = -1 \tag{26}$$

Thus we see that for the fermions in the $\mathbf{5}^*$ and $\mathbf{10}$ representations the anomalies cancel themselves,

$$A(\mathbf{5}^*) + A(\mathbf{10}) = 0 \tag{27}$$

The GUT with a simple group has by definition a single coupling constant. The possibility of different subgroup couplings is realized through SSB, when the X and Y gauge bosons achieve masses and thus contributes differently to radiative corrections. Let us try to extract information about the Weinberg angle from the Georgi-Glashow model. At a scale much above $M_{X,Y}$ we have

$$g_3 = g_2 = g_1 = g_5 \tag{28}$$

where the subscripts refer to strong, weak, electromagnetic and $SU(5)$ respectively. The Weinberg angle is defined as,

$$Sin^2(\theta_W) = \frac{g'^{\,2}}{g^2 + g'^{\,2}} \tag{29}$$

where g and g' are the coupling constants of the $\mathbf{A}'_\mu$ and $\mathbf{B}'_\mu$ gauge bosons in the electroweak theory. With the properly normalized generators of $SU(5)$ we have the identification $g' = -(3/5)^{1/2}\, g_1$ and $g = g_2$. Then from (28) and (29) we have the result that above the GUT scale,

$$Sin^2(\theta_W) = \frac{g'^{\,2}}{g^2 + g'^{\,2}} = \frac{3/5}{3/5 + 1} = \frac{3}{8} \quad (k^2 > M_X) \tag{30}$$

Using renormalization group equations we can study how the coupling constants flow from this scale down to the electroweak scale and this gives [5],

$$Sin^2(\theta_W) \simeq 0.21 \quad (k^2 \sim M_Z) \tag{31}$$

This is surprisingly close to the measured value at the scale of M_Z which is $Sin^2(\theta_W) = 0.23120(15)$ [7].

The Georgi-Glashow model also has many other interesting phenomenological implications that we do not discuss here. One interesting conjecture pertains to the baryon asymmetry in the universe as resulting from the presence of the X, Y lepto-quark gauge bosons and CP violation in the model. Another fascinating topic is the origin of flavor and mixing angles in the model. The interested reader is referred to the excellent review of GUTs by Langacker [2] and [5].

5 Conclusions

We have seen that the grand unified theory introduced by Georgi and Glashow based on the group $SU(5)$ displays many attractive features and attempts to relate some of the arbitrary features of the standard model : It explains charge quantization and relates the quark and lepton charges, the anomalies automatically cancel and it is found that the prediction for the electroweak angle is quite close to the measured value. In spite of its many successes and features the current view point is that a GUT based on $SU(5)$ alone is incomplete. One reason is the observation that neutrinos carry small masses and the strong indication that there might be right handed Majorana neutrinos (See for example [8]). It is not very straight forward to introduce right handed neutrinos in the $SU(5)$ model. This has renewed interest in GUTs based on the gauge group $SO(10)$ where the spinor representation can readily accommodate the 16 left handed fields and also GUTs based on the exceptional group E_6 which is motivated by string theories [2]. But in most of the models $SU(5)$ does appear as an intermediate step after SSB. Thus in the current context the Georgi-Glashow model must be viewed as not an incorrect but as an incomplete theory of the SM interactions [9].

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